v.2, no.3
Mathematics
QA
1
.M426

THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

Volume II March, 1910 Number 3

CONTENTS

Editorial 81
Some Common Errors in Elementary Algebra 84
Approximate Values of π. By WILFRED H. SHERK 87
Some Suggestions in the Teaching of Geometry. By ISAAC J. SCHWATT . . . 94
Mathematics for Service. By ERNEST R. VON NARDROFF 116
New Books 127
Notes and News 129

Published Quarterly by the

ASSOCIATION OF TEACHERS OF MATHEMATICS
FOR THE MIDDLE STATES AND MARYLAND

LANCASTER, PA., SYRACUSE, N. Y.
NEW YORK, N. Y., PHILADELPHIA, PA.

THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

EDITOR

W. H. METZLER, Ph.D., F.R.S.E.
Syracuse University, Syracuse, N. Y.

ASSOCIATE EDITORS

EUGENE R. SMITH, M.A.
Polytechnic Preparatory School, Brooklyn, N. Y.

JONATHAN T. RORER, Ph.D.
Central High School, Philadelphia, Pa.

THE MATHEMATICS TEACHER is published quarterly—September, December, March and June—under the auspices of the Association of Teachers of Mathematics for the Middle States and Maryland.

The annual subscription price is $1.00; single copies, 35 cents.

Remittances should be sent by draft on New York, Express Order or Money Order, payable to The Mathematics Teacher.

THE MATHEMATICS TEACHER,
41 NORTH QUEEN STREET, LANCASTER, PA.
or SYRACUSE, N. Y.

Ready---March 10, 1910

PLANE TRIGONOMETRY

BY

DR. FLETCHER DURELL,

Head of the Mathematical Department of the Lawrenceville School, and author of Durell's Geometry.

Meets the latest requirements of the large Universities. Embodies newest systems of practice in use in the United States Navy Department and by general Engineering Companies. In solution of triangles the degree is divided both sexagesimally and decimally. For further information address

---

CHARLES E. MERRILL CO.,

44-60 East 23rd St., New York City

Publishers of School and College Text-Books

To the Friends of Dr. Edward Brooks

DR. EDWARD BROOKS, principal of the First Normal School of Pennsylvania, Superintendent of the schools of Philadelphia, a Founder of the National Educational Association, author of a Self-living Series of Mathematics and of other educational books, is well known to all teachers throughout the United States.

Dr. Brooks has now retired from public educational work, but he continues his authorship with unabated enthusiasm. **The Normal Standard Algebra** (1909) is his latest product. It is a book absolutely clear, logical, and teachable, which in compact form embodies all the best modern thought on methods in Algebra. It is a new mathematical masterpiece fresh from the vigorous mind of the master. The many personal friends of Dr. Brooks and the thousands who are friends only through his writings will welcome this new book with joy, and will find in it the genius touch which has made its author famous.

CHRISTOPHER SOWER COMPANY

PUBLISHERS

614 Arch Street **Philadelphia**

Six Teachers of One Mind

"Fullest Measure of Mental Training"

I know of no text-book on the subject so well calculated to give the student a knowledge of Geometry while securing the fullest measure of mental training.

M. T. DANA,
Professor of Mathematics, State Normal School, Fredonia, N. Y.

"Gives the Best Results"

I have several different texts in Geometry in the Medford High School and I have no hesitancy in saying that "Shutts' Geometry in the Suggestive Method" has given the best results.

F. T. WINGATE,
Instructor in Mathematics, High School, Medford, Mass.

"Fitted to the Ordinary Pupil"

We are very much pleased with "Shutts' Geometry in the Suggestive Method" and are obtaining good results. It starts the pupils right, as it requires them to do their own thinking from the beginning. The work throughout is fitted to the ordinary pupil.

MOUNT ST. JOSEPH COMMERCIAL HIGH SCHOOL
Hartford, Connecticut.

"Used in Boston"

"Shutt's Geometry in the Suggestive Method" is used in the Mechanic Arts High School of Boston. I entertain a high opinion of the general plan and development of the book.

WILLIAM FULLER
Head of the Department of Mathematics, Mechanic Arts High School, Boston

"Makes Pupils Reason Rather Than Remember"

I have used "Shutt's Geometry" for the past two years and I believe that its suggestive method has been very effective in causing the pupils to reason rather than remember.

LOUISE CARR,
Instructor in Geometry, Wellesley High School, Wellesley, Mass.

"Has Favorable Commendation"

I made the acquaintance of "Shutt's Geometry in the Suggestive Method," a number of years ago, and year by year its distinctive features have received favorable comment in my "Method" work in the normal school.

A. W. HUSTED
Professor of Mathematics in the Albany, N. Y. State No

ATKINSON, MENTZER & GROVER, - - - - Boston, New York, Chicago, Dallas

TEXT-BOOKS OF TRIGONOMETRY TABLES OF LOGARITHMS

By EDWIN S. CRAWLEY

Professor of Mathematics in the University of Pennsylvania

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY. New and revised edition, vi and 186 pages, 8vo. **Price, $1.10.**

THE SAME, WITH FIVE-PLACE TABLES (as below), half leather. **Price, $1.50.**

This book is intended primarily for college use, but is used in many secondary schools also.

SHORT COURSE IN PLANE AND SPHERICAL TRIGONOMETRY. 121 pages, 8vo. **Price, $0.90.**

THE SAME, WITH FOUR-PLACE TABLES. **rice, $1.00.**

THE SAME, WITH FIVE-PLACE TABLES. Price, $1.25.

This book is intended primarily for use in secondary schools, but many colleges, in which the time allotted to trigonometry is restricted, have adopted it.

TABLES OF LOGARITHMS, to five places of demicals. Seven tables, with explanations. xxxii and 76 pages, 8vo. **Price, $0.75.**

These tables are much better arranged than most of the tables prepared for school use.

Orders, and requests for books for examination with a view to introduction, should be directed to

EDWIN S. CRAWLEY, University of Pennsylvania, Philadelphia

N. B.—In all cases specify by full title which book is desired.

The Nineteenth Summer Session of

Cornell University

JULY 6 TO AUGUST 16

offers a dozen different courses in Mathematics, including two of distinctly pedagogic character which deal directly with the problems of the school teacher. There is instruction in twenty-five other departments. The fine equipment of the university together with the unequaled opportunities for field work in science are attracting large numbers of teachers every summer. Full information may be had by application to

THE REGISTRAR

Cornell University ITHACA, N. Y.

REVIEW QUESTIONS

ALGEBRA PLANE GEOMETRY

CHEMISTRY SOLID GEOMETRY

PHYSICS TRIGONOMETRY and LOGS

Compiled from recent College Entrance Examinations. Six Pamphlets. Price, 40 cents each. Sample copy half price. Teacher's copy free on adoption for class use.

NOW IN USE IN MANY OF THE BEST SCHOOLS OF THE COUNTRY

Address FRANKLIN T. JONES, University School, Cleveland, Ohio

UNIVERSITY OF PENNSYLVANIA

SUMMER SCHOOL

TERM: JULY 5TH TO AUGUST 12TH.

Both Undergraduate and Graduate Courses in the following subjects, leading to the degrees of A.B., B.S., M.A., and Ph.D. Architecture, Botany, Chemistry, Economics, English, French, Geography, German, Greek—both Classical and New Testament, Hebrew, History, Italian, Latin, Mathematics, Medicine, Music, Pedagogy, Physical Education, Physics, Psychology, Public Speaking, Spanish and Zoology.

Special Courses for Architects, Elementary School Teachers, Kindergartners, High School Teachers, Teachers in charge of School Playgrounds and Gardens, Instructors in Physical Education, Social Workers, Teachers of Backward Children, Physicians and Clergymen.

Thirteen Courses in Mathematics, including Algebra, Geometry, Plane and Spherical Trigonometry, Analytic Geometry, Differential and Integral Calculus and The Teaching of Elementary Mathematics.

Psychological Clinic, School of Observation, Architectural Drawing-rooms, Botanical and School Gardens, Chemical, Physical, Medical and Biological Laboratories, University Museum, Library, Gymnasium, Swimming-pool and Athletic Grounds.

For circular and information concerning special railroad and dormitory rates, address

A. DUNCAN YOCUM, Director of the Summer School
Box 8, College Hall, University of Pennsylvania,
PHILADELPHIA, PA.

COTRELL & LEONARD

ALBANY, N. Y.

Makers of

CAPS
GOWNS AND
HOODS

To the American Colleges and Universities
From the Atlantic to the Pacific

CLASS CONTRACTS A SPECIALTY

Teachers College
Columbia University
New York

Summer Session: Opens on July 6, 1910, and closes on August 16th.

Professor David Eugene Smith will offer two courses, one on the history and teaching of algebra, and one on the history and teaching of geometry. Numerous other courses in mathematics.

The Academic year 1910–1911. Several new courses for teachers of mathematics.

Circulars sent on application. Address

THE SECRETARY OF TEACHERS COLLEGE
Columbia University NEW YORK CITY

"CHAUTAUQUA"

Means These Three Things. Which Interests You?

A System of Home Reading

Definite results from the use of spare minutes. Classical year now in progress. *Ask for C. L. S. C. Quarterly.*

A Vacation School

Competent instruction. Thirteen Departments; Over 2500 enrollments yearly. The best environment for study. Notable lectures. Expense moderate. July and August. *Ask for Summer School Catalog.*

A Metropolis in the Woods

All conveniences of living, the pure charm of nature, and advantages for culture that are famed throughout the world. Organized sports, both aquatic and on land. Professional men's clubs. Women's conferences. Great lectures and recitals. July and August. *Ask for Preliminary Quarterly.*

Chautauqua Institution, Chautauqua, New York

20 Reasons Why You Should Purchase

The No. 12 Model

HAMMOND

1. Visible Writing.
2. Interchangeable Type.
3. Lightest Touch.
4. Least Key Depression.
5. Perfect and Permanent Alignment.
6. Writing in Colors.
7. Least Noise.
8. Manifolding Capacity.
9. Uniform Impression.
10. Best Mimeograph Work.
11. Any Width of Paper Used.
12. Greatest Writing Line.
13. Simplicity of Construction.
14. Greatest Durability.
15. Mechanical Perfection.
16. Back Space Attachment.
17. Portability.
18. Least Cost for Repairs.
19. Perfect Escapement.
20. Beauty of Finish.

WRITE FOR CATALOG

The Hammond Typewriter Company

69th to 70th Streets, East River

NEW YORK, N. Y.

WHEELER'S ALGEBRA

First Course in Algebra

By ALBERT HARRY WHEELER, Teacher of Mathematics in the English High School, Worcester, Mass.

Half leather, $1.15.

This modern book includes:

Three thousand mental exercises

The largest number of graded written examples

The best development of algebra from arithmetic

A lucid and early treatment of graphs

A through system of numerical checks

Proofs and reasons marked for omission as class work at the discretion of the teacher

Problems in Physics

THE SAME. BRIEF EDITION. (As far as Quadratics.)

Cloth, 95 cents.

Algebra for Grammar Schools

By A. H. WHEELER. Cloth, 50 cents.

LITTLE, BROWN & CO.

34 Beacon St., Boston **378 Wabash Ave., Chicago**

Just Published

PLANE GEOMETRY

DEVELOPED BY THE

SYLLABUS METHOD

By EUGENE RANDOLPH SMITH, A.M.

Head of the Department of Mathematics, Polytechnic Preparatory School, Brooklyn, N. Y. Formerly Head of the Department of Mathematics, Montclair High School, Montclair, N. Y.

Price, 75 cents

This is the only complete and thoroughly worked-out plan of teaching geometry by the Syllabus Method. It makes the study more profitable. It makes the pupil do the thinking, not the teacher or the author. It teaches pupils how to discover the proofs for themselves instead of memorizing those given in the text-book. It gives the maximum of mental training with the minimum waste of energy.

Except for a few specimen proofs and some necessary hints, the demonstrations are omitted, but before the work is undertaken, propositions having any difficulty are fully discussed in class with the teacher, who guides the investigation by questions wh ch do not imply their own answers. The pupil is made familiar with the laws of logic used in plane geometry, and understands the "what" and "why" of each step. Consequently he has confidence in his own powers from the beginning, he progresses with great rapidity, and is able to do a large amount of original work without effort.

Harvard University

I have looked over the copy of Smith's Plane Geometry which you sent me, and am very favorably impressed by it. It is the best elementary plane geometry in English with which I am acquainted. If I had to teach plane geometry, I should certainly try the book.

JAMES K. WHITTEMORE,
Assistant Professor of Mathematics.

AMERICAN BOOK COMPANY

New York **Cincinnati** **Chicago** **Boston**

THE MATHEMATICS TEACHER

EDITED BY

W. H. METZLER

ASSOCIATED WITH

EUGENE RANDOLPH SMITH

JONATHAN T. RORER

Volume I

SEPTEMBER, 1908—JUNE, 1909

THE MATHEMATICS TEACHER

SYRACUSE, N. Y.

THE MATHEMATICS TEACHER

EDITED BY
W. H. METZLER

ASSOCIATED WITH
EUGENE R. SMITH JONATHAN T. RORER

VOLUME II MARCH, 1910 NUMBER 3

EDITORIAL.

Stimuli.

All teachers have heard discussions, if they have not themselves taken part in them, as to the advantage or disadvantage of prizes and other rewards for pupils in connection with school work. These discussions are perhaps not as common today as they were a few years ago and a good many of them no doubt failed to hit the mark.

Stimuli, to be of permanent value, must come from within and not from without. The familiar figures of the horse so full of life that it has to be held back without a whip in sight, and the one that has to have the whip applied every few steps to make any progress at all, are familiar examples of the two types. The one horse lives, while the other merely exists. For both there is exercise. For the one there is interest and enjoyment in the exercise, for the other there is neither.

Mental exercises with interest and enjoyment is one of the great secrets of school life. Usually the very exercise itself furnishes the enjoyment, although sometimes the enjoyment comes through seeing a value, to the person involved, in acquiring that which the exercise gives. Every healthy person likes to exercise physically for the reason that the exercise gives enjoyment, or because he believes that by exercising he will become more healthy or keep his health longer. So in mental life we exercise the mind because we enjoy it or because we feel it must be done to make it stronger and to keep it so. Teachers should remember that things are not necessarily interesting

because they are easy. One piece of work may be uninteresting because it is easy and another may be interesting because it is hard.

Helpfulness.

To be useful and helpful to others is an art that every teacher should acquire. To be helpful to others is not only to be of service to them but it is to be of service to ourselves as well. We cannot help others without benefiting ourselves.

Every member of our association can be of great use to the association. You can interest yourself in the affairs of the association, and though you may not feel like writing an article, you may have some ideas or suggestions that could be expressed in a few sentences or paragraphs. If so send them along and the editor will make use of them from time to time.

Again, every member of the association can be of use in extending the influence of the association by getting others to become members. This will be helping both the association and the new members. If you believe the association is a benefit to you, then why not make it a benefit to every teacher of mathematics within our territory? You know that THE MATHEMATICS TEACHER is stimulating and helpful to you in your work, and you should be interested in bringing it to the attention of every teacher of mathematics whom you know. There are a thousand or more teachers in our territory who should be members, but are not likely to be unless you go after them. If every member would bring in just one during the year a great deal would be accomplished. Some members have already sent in as many as four new names and are looking around for more. Will you not do your part to make the association what it aims to be and what it should be—a help to every teacher?

Let us see to it that the following indictment quoted from a recent letter to the editor from a prominent member of our association is to be no longer true in our territory:

"I do not wish to say anything too strong, but I feel that there is not a more indifferent class of people than teachers are. All that the most of them do is mark the time. I feel that all the teachers who are engaged in the teaching of mathematics in the Middle States and Maryland ought to become members

and read THE MATHEMATICS TEACHER, and try to improve themselves. It is hard to think that even the few hundred members which we have show so little interest in the association. I am led to make this statement by the poor attendance of our last meeting. I understand that this apathy is not peculiar to New York or Philadelphia, but the same condition shows itself all over the country. A friend of mine, a superintendent of one of the largest schools of this union, told me that it is the hardest thing in the world to get teachers to improve themselves. They all feel that they are poorly paid and ill treated, and that they do more than they really ought to do. I sometimes feel that if our educational leaders, instead of continually telling the teachers of the great work they are doing and how efficient and noble they are, would be frank enough with them and tell them how poorly, on the whole, the average teacher is equipped for his work and the little good he is able to do, would stimulate them to greater effort.

"Most of the articles which I see either in the American or European pedagogical magazines are all lacking the important function of stimulating the teacher to greater effort. It is sad to think that teaching seems to be the only profession in which mediocracy and incompetency can subsist the longest."

SOME COMMON ERRORS IN ELEMENTARY ALGEBRA.*

(1) $$3(2x-4m)=6x-4m.$$

(2) $$(a)\qquad 1^3=3,$$
$$(b)\quad (-3)^2=-9.$$

(3) $$(x+a)^2=x^2+a^2.$$

(4) $$(a+x)(b+x)=ab+x^2.$$

(5) $$\frac{x^5-y^5}{x-y}=x^4+y^4.$$

(6) $$a(c+d)+b(c-d)=(a+b)(c+d)(c-d).$$

(7) $$a^3+b^3=(a+b)(a^2\pm 2ab+b^2).$$

(8) $$(a+b)^3=(a+b)(a^2-ab+b^2).$$

(9) $$\frac{x+a}{2y+a}=\frac{x}{2y}.$$

(10) $$\frac{2x+y}{2x+y}=0 \quad \text{and} \quad \frac{x-x}{2a}=\frac{1}{2a}.$$

(11) $$a+\frac{b}{c}=ac+b.$$

(12) $$\frac{a}{b}\times c=\frac{ac}{bc}.$$

(13) $$\frac{a}{b}-\frac{b-c}{b}=\frac{a-b-c}{b}$$

and

$$a-\frac{b-c+d}{e}=a-\frac{b}{e}-\frac{c}{e}+\frac{d}{e}.$$

(14) $$\frac{x-\frac{b}{c}}{d}=\frac{x-b}{cd}.$$

* See the report of the Philadelphia Section, page 73, of last issue.

(15) $$\frac{a}{a-b} = -\frac{a}{a+b}.$$

(16) $$\frac{x}{5} = 30 \quad \therefore x = 6.$$

(17) $$x = \tfrac{1}{2} \quad \therefore \frac{6}{x} = 3.$$

(18) $$ax = b, \ \therefore x = \frac{a}{b}.$$

(19) $$abx = c, \quad \therefore \ x = c - ab.$$

(20) $$\frac{x}{3} = a, \quad \therefore x = -3a.$$

(21) $$\frac{x-a}{b} = 0, \quad \therefore \ x - a = b.$$

(22) $$y = -1, \quad x - 3y = 8,$$
$$\therefore \ x - 3 = 8.$$

(23) $$\frac{x}{a} + \frac{y}{b} = 1, \quad bx + ay = 1.$$

(24) $$x = 2, \quad 4x = 8x. \quad \text{(In proofs.)}$$

(25)
$$\begin{array}{r} abx + cdy = a \\ abx + \ \ y = d \\ \hline cdy - y = a - d \\ y = \dfrac{a-d}{cd} \end{array} \quad \text{or} \quad \begin{array}{r} abx + cdy = a \\ abx + \ \ y = d \\ \hline cd = a - d \end{array}$$

(26) $$(a^2)^3 = a^5, \quad (a^2)^3 = a^8.$$

(27) $$(a) \ a^{-3} = a^{\frac{1}{3}}, \quad (b) \ a^{\frac{1}{3}} = a^3, \quad (c) \ a^{\frac{1}{3}} = \frac{1}{\sqrt{a^3}},$$
$$(d) \ 3a^{-2} = \frac{1}{3a^2}, \quad (e) \ 3^{-1} = \frac{1}{-3}, \quad (f) \ a^0 = a.$$

(28) $$(a) \ \sqrt{\frac{x}{a}} = a\sqrt{ax}, \quad (b) \ \sqrt{45} = 5\sqrt{3}, \quad (c) \ \frac{1}{\frac{2}{3}\sqrt{3}} = \tfrac{3}{2}\sqrt{3}.$$

$$(29)\quad \begin{array}{ll} (a) & (\sqrt{3x-1}-2)^2=3x-1\pm 4, \\ (b) & 2\sqrt{a+b}=5,\ \therefore\ 2(a+b)=25, \\ (c) & \sqrt{x}=4,\ \therefore\ x=2. \end{array}$$

$$(30)\quad ax^2+bx=c,\quad x(ax+b)=c,\quad x=\frac{c}{ax+b}.$$

$$(31)\quad x^2-5x=0,\quad x-5=0,\quad x=5 \text{ and } x^2-4x+4=25.$$

$$(32)\quad x(x-4)=3,\ \therefore\ x=3,\quad x-4=0,\quad x-2=5.$$

"How may we expect to develop individualism by our present system? We do not do so. It is a fact that individualism exists, but it is in spite of our school system, and not because of it. What opportunity do we give our children to learn of the beautiful in life when we house them in jail-like structures? How are we developing manhood when we keep boys cooped up like animals and treat them almost like criminals? We want to give them a chance to work hard and to play hard, to give knocks and to take knocks, to get some of the discipline of life at the very beginning, so that it will not fall upon them suddenly when they are totally unprepared.

"I suppose that Lord Cromer is accounted one of the most 'practical' men in the world. For thirty years he was the great British proconsul in Egypt, and he ruled the most ancient civilization of the world as it had never been ruled before. He took it when it was at its lowest ebb, and brought it to a state of prosperity never before equalled. The British nation has rightly honored and rewarded him. Because he has accomplished so much, we have naturally looked upon him as a sort of Bismarck or Clive, a man of 'blood and iron.' Yet in his book, recently published, he tells us how all this was accomplished without the drawing of a sword. He ruled and governed by mere force of character and through the inspiration of lofty ideals.

"It is deplorable that there is nothing in our public schools calculated to bring out such latent powers in our boys. Many grand men and women come out of our public schools, it is true, but small praise is due the schools for the result."—From "What is Wrong with Our Public Schools," by JOSEPH M. ROGERS, in February *Lippincott's*.

APPROXIMATE VALUES OF π.*

By Wilfred H. Sherk.

Last spring I asked one of my classes to find the approximate value of π after the manner presented in Wentworth's "Plane Geometry." Only one pupil had the perseverence to do the great amount of accurate work necessary to get the result correct to five places of decimals. This fact led me to seek a less laborious way. I tried finding the area of a circle with unit radius. Area equals π. I did this by methods which could be applied in finding the areas of closed plane surfaces other than the circle, with the hope of finding a method so simple and accurate that it could be presented to high school pupils and with the hope of thus furnishing the boys and girls with a mathematical tool which should be of use in subsequent mathematical work. My hope was not realized, yet the various formulæ employed were interesting.

I. Rectangular Method.

The equation, $x^2 + y^2 = 1$, was assumed. The range, $0 - 1$, was divided into one hundred equal parts, and the ordinates corresponding to the abscissas of the points of division were computed. Then,

A. The sum of the series of rectangles inscribed in a quadrant of the circle was computed, and

B. The sum of the series of rectangles circumscribed about a quadrant was obtained.

Four times either sum gave an approximate value of π. Consideration of the definite integral suggested this method.

II. Trapezoidal Rule.

The upper ends of the ordinates computed as above were connected by straight lines thus forming a series of inscribed trapezoids. If $y_0, y_1, y_2, y_3 \cdots y_n$ are the ordinates of the points of division of the range, $0 - 1$, and h represents the length of one of these divisions the areas of the trapezoids are, $\frac{1}{2}h$

* Read before the Rochester Section.

$(y_0 + y_1)$, $\frac{1}{2}h\,(y_1 + y_2)$, $\frac{1}{2}h\,(y_2 + y_3)$, $\cdots$, the sum of which gives

$$\text{Area} = h\,(\tfrac{1}{2}\,y_0 + y_1 + y_2 + y_3 \cdots + y_{n-1} + \tfrac{1}{2}\,y_n).$$

Writing only the coefficients, as is customary in writing such formulæ,

$$\text{Area} = h\,(1/2 + 1 + 1 + 1 + 1 + 1 \cdots + 1 + 1/2).$$

This formula, of course, gave for π a value which was the mean of the results given by A and B of I.

III. Cotes's Method.

The values of π thus far obtained were not correct through two places of decimals. Of course more accurate results could have been obtained by increasing the number of equal parts into which the range, 0 — 1, was divided, but 100 was as many as my time and patience would allow. It became evident then that if more accurate results were to be obtained, the upper boundary of the little parts into which the quadrant was divided would have to be something other than straight lines. This consideration led to an examination of a number of rules for approximating the area included between two ordinates, a given curve, and the axis of X:

A. Simpson's 1/3 rule, or the parabolic rule.

B. Simpson's 3/8 rule.

C. Weddle's rule.

These and several other rules were found to be special cases of a more general formula attributed to Roger Cotes, a contemporary of Newton and the editor of Newton's "Principia." Cotes made use of Lagrange's well known interpolation formula the derivation of which follows.

Suppose $f(x)$ is finite, continuous, and single valued in the range $a - b$, and that it is desired to replace $f(x)$ by another function of x that shall differ from $f(x)$ little at will. Plot the curve of $y = f(x)$, and divide the range, $a - b$, into n equal parts whose abscissas are $x_0, x_1, x_2, x_3, x_4, \cdots x_n$. Compute the corresponding ordinates, $y_0, y_1, y_2, y_3, y_4, \cdots y_n$. We now have $(n + 1)$ points of $y = f(x)$, given by their coordinates. The problem now is to find a function of x, call it

$F(x)$, whose curve shall pass through these $(n+1)$ points. $F(x)$ has the form,

$$F(x) = S_0 + S_1x + S_2x^2 + S_3x^3 + \cdots S_nx^n,$$

in which the S's are to be determined. One could write $n+1$ equations of the form,

$$y_r = S_0 + S_1x_r + S_2x_r^2 + S_3x_r^3 + \cdots S_nx_r^n,$$

solve for the S's and substitute their values in

$$F(x) = S_0 + S_1x + S_2x^2 + S_3x^3 + \cdots S_nx^n,$$

and thus get $F(x)$.

A procedure which leads to more general results, however, is that followed by Lagrange. He assumed the identity,

$$Q(x) \equiv (x - x_0)(x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n).$$

$F(x)/Qx$ is evidently a proper fraction and can be broken up into partial fractions. To do this make use of a rule thus stated by Byerly, "To find the numerator corresponding to a linear denominator, $X - a$, we have simply to strike that factor out of the denominator of the given fraction, and then substitute a for x in the result." The fraction corresponding to the denominator $x - x_r$ is then

$$\frac{\left[\dfrac{\dfrac{F(x)}{Q(x)}}{x - x_r}\right]_{x=x_r}}{x - x_r}$$

and if the sum of such partial is taken,

$$\frac{F(x)}{Q(x)} = \sum_{r=0}^{r=n}\left\{\frac{\left[\dfrac{\dfrac{F(x)}{Q(x)}}{x - x_r}\right]_{x=x_r}}{x - x_r}\right\} = \sum_{r=0}^{r=n}\left\{y_r\frac{\left[\dfrac{1}{\dfrac{Q(x)}{x - x_r}}\right]_{x=x_r}}{x - x_r}\right\}$$

whence

$$F(x) = \sum_{r=0}^{r=n}\left\{\frac{\dfrac{Q(x)}{x - x_r}}{\left[\dfrac{Q(x)}{x - x_r}\right]_{x=x_r}}\right\}$$

and

$$F(x)=\sum_{r=0}^{r=n}\left\{ y_r \frac{\begin{array}{r}(x-x_0)(x-x_1)(x-x_2)\cdots \\ (x-x_{r-1})\,\#\,(x-x_{r+1})\cdots(x-x_n)\end{array}}{\begin{array}{r}(x_r-x_0)(x_r-x_1)(x_r-x_2)\cdots \\ (x_r-x_{r-1})\,\#\,(x_r-x_{r+1})\cdots(x_r-x_n)\end{array}}\right\}.$$

Suppose now that the range is from 0 to a

$$x_0=0,\ x_1=\frac{a}{n},\ x_2=\frac{2a}{n},\ x_3=\frac{3a}{n},\ \cdots,\ x_n=\frac{na}{n}$$

whence by substitution and multiplying numerator and denominator by n^n,

$$F(x)=\sum_{r=0}^{r=n}\left\{ y_r \frac{\begin{array}{l}nx(nx-a)(nx-2a)\cdots\{nx \\ \quad -(r-1)a\}\#\{nx-(r+1)a\}\cdots(nx-na)\end{array}}{\begin{array}{l}nx_r(nx_r-a)(nx_r-2a)\cdots\{nx_r \\ \quad -(r-1)a\}\#\{nx_r-(r+1)a\}\cdots(nx_r-na)\end{array}}\right\}.$$

That is,

$$F(x)=y_0\frac{\#(nx-a)(nx-2a)\cdots(nx-na)}{\#(0-a)(0-2a)\cdots(0-na)}$$

$$+y_1\frac{nx\#(nx-2a)\cdots(nx-na)}{a\#(a-2a)\cdots(a-na)}$$

$$+y_2\frac{nx(nx-a)\#(nx-3a)\cdots(nx-na)}{2a(2a-a)\#(2a-3a)\cdots(2a-na)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+y_n\frac{nx(nx-a)(nx-2a)\cdots\{nx-(n-1)a\}\#}{na(na-a)(na-2a)\cdots\{na-(n-1)a\}\#}.$$

(In these formulæ the symbol # indicates the position occupied by a factor which has been cancelled.)

The locus of this equation passes through $(n+1)$ points of the locus of $y=f(x)$ within the range, $0-a$. It is to be noticed that the coefficients of the y's are independent of the given function, $f(x)$, and so can be computed once for all for various values of a and n. Herein lies the beauty and efficiency of Lagrange's interpolation formula.

Now the definite integral, $\int_0^a F(x)dx$, is graphically represented by the area included between the locus of $y = F(x)$, the axes and the ordinate, y_a. Cotes seized upon this fact, and after replacing $f(x)$ by $F(x)$ which differed from $f(x)$ little at will, he found $\int_0^a F(x)dx$. This gave him an area which differed from the area included by $y = f(x)$, the axes, and y_a as little as he chose to make it, the amount of difference depending only upon the patience and accuracy of the computer.

A. Simpson's 1/3, or the Parabolic Rule.

Following Cote's method, let $n = 2$ in Lagrange's formula and get the definite integral,

$$\int_0^a F(x)dx = \frac{a}{6}(y_0 + 4y_1 + y_2).$$

If h represents one of the equal parts into which the range $0 - a$, is divided, then $a = 2h$ and

$$\int_0^a F(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now if any range, $0 - c$, is divided into a number of equal parts each equal to a and the formula just derived is applied in succession to each range, and the definite integrals summed,

$$\int_0^c F(x)dx = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{m-1} + y_m),$$

or

$$\int_0^c F(x)dx = \frac{h}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 4 + 1).$$

This is Simpson's 1/3 rule, also called the parabolic rule because its derivation is usually made to depend upon the properties of the parabola.

B. Simpson's 3/8 Rule.

If in Lagrange's formula we let $n = 3$ and proceed as in obtaining the parabolic rule,

$$\int_0^c F(x)dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + \cdots + 3y_{m-1} + y_m),$$

or

$$\int_0^c F(x)dx = \frac{3h}{8}(1 + 3 + 3 + 2 + 3 + 3 + \cdots + 3 + 1).$$

This formula is known as Simpson's 3/8 rule.

C. Weddle's Rule.

If we put n equal to 6 and proceed as above,

$$\int_0^a F(x)dx = \frac{3h}{10}\left[\tfrac{41}{42}y_0 + \tfrac{216}{42}y_1 + \tfrac{27}{42}y_2 + \tfrac{272}{42}y_3 + \tfrac{27}{42}y_4 + \tfrac{216}{42}y_5 + \tfrac{41}{42}y_6\right].$$

Mr. Thomas Weddle called this

$$\int_0^a F(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6),$$

or

$$\int_0^a F(x)dx = \frac{3h}{10}(1 + 5 + 1 + 6 + 1 + 5 + 1),$$

or in general

$$\int_0^c F(x)dx = \frac{3h}{10}(1 + 5 + 1 + 6 + 1 + 5 + 2 + 5 + 1 + 6 + 1 + 5 + \cdots + 5 + 1).$$

Cotes carried these formulæ out to the case in which n equals ten. The results, however, are too cumbersome to be useful.

The value of π obtained by these various methods were disappointing. They were as follows:

By circumscribing rectangles	3.1586
By inscribing rectangles	3.1186
By Trapezoidal rule	3.1386
By Parabolic rule	3.1547
By Simpson's 3/8 rule	3.1385
By Weddle's rule (error)	

Of course there are many other ways of approximating the value of π. Perhaps as simple and rapid a way as any is by the method of series taught in differential calculus. However after making myself weary computing the value of π, I took a crude

circular disk, an ordinary scale, and a piece of paper, measured the circumference and the diameter of the disk, divided one by the other, and obtained a result that was surprisingly accurate. I concluded then that until I had more light I would not require my pupils in elementary geometry to compute the value of π. They must show that the ratio of the circumference of a circle to its diameter is constant, but the value of that ratio they may measure.

LAFAYETTE HIGH SCHOOL,
BUFFALO, N. Y.

Fundamentally, the trouble with our system of public education is that children learn a little about a great many things, without gaining much really definite knowledge of anything which is likely to stand them in good stead in later life; and, what is equally bad, they do not acquire methods of accurate thinking. This criticism applies to education in every branch, and has been true from the beginning because our theories have been wrong. But just now it is peculiarly true that our public school system has defects which are so palpable, and for which remedies are so easily found, that we should no longer permit things to remain as they are.

It cannot be too strongly impressed upon the American people that the so-called "three R's" are not grounded in the youthful minds of this generation as they should be. Reading, Writing, and Arithmetic are the tools of the human mind. Without them almost nothing of an intellectual sort can be accomplished. We ought to put the very best tools in the hands of our children, and they should be kept in the best possible condition. Professor Barrett Wendell, of Harvard University, recently told in a lecture how one of his brightest students did not know the letters of the alphabet in their order, and in consequence was much hampered in the use of a dictionary. When it comes to writing, we have retrograded. At various times much stress has been laid on various systems of penmanship which have been in turn adopted and discarded, with the result that few children write so well as they should, or so well as did their forebears at the same age. As to composition, most children are befogged at the simplest test.—JOSEPH M. ROGERS in January *Lippincott's*.

SOME SUGGESTIONS IN THE TEACHING OF GEOMETRY.

By Isaac J. Schwatt.

The average student seems to have less understanding of geometry than of any other of the elementary mathematical disciplines. Even calculus—of which Luebsen, one of the ablest writers of mathematical text-books, has said, that out of ten who study the subject only two understand it and still fewer are able to apply it—is not excepted. This is in part due to the extreme difficulty of the subject of geometry, the age at which the study of it is usually attempted, and to the relatively small amount of time alloted to it.

Instruction in geometry ordinarily consists in the study of the theorems and the proofs given in the text with comparatively little work in the solution of problems of construction. In this respect instruction in geometry differs from that in algebra. The latter consists mostly in the working of examples and problems; very little theory is given. One need only compare the examination questions in these two subjects set by the schools and for admission to colleges.

Under the present system of marking, it is possible for a candidate to pass almost any of the examination papers in geometry which I have seen, if he is able to prove the theorems as given in most text-books, without having acquired facility in solving constructions. I know of a case where the insertion in the entrance examination paper in algebra for admission to a university of one or two questions pertaining to the theory of the subject, brought serious protest from teachers who prepare students for that institution. To follow the general trend and expediency, it was deemed advisable to omit such questions in subsequent papers.

While a pupil may pass an examination in geometry by memorizing theorems and proofs, to pass the examination in algebra he must understand its processes and acquire facility in their use. Even if the examination paper in algebra does not contain any

questions on the theory of the subject, the data and the figures of the examples on it are different from any he has seen before. The working of constructions develops in the mind of the learner just such faculties and such powers as he will be in need of whatever his vocation or occupation in life.

One of the first requisites for success in life, whether in the performance of professional duties or in commercial enterprise, consists in the ability to bring all the possibilities of a situation instantaneously before the mind and to select the one which will give the required result. Take for instance the work of the physician. From his knowledge of the human body in its healthy state and from his experience with it in a pathological state, the physician must make combinations and apply them to the case to be diagnosed in such a manner as will enable him to decide wherein the deficiency of the human mechanism lies. Exactly such training is furnished by the working of constructions. The work in constructions calls for continuous concentration and uninterrupted attention, for the utmost patience, for the quality of not being discouraged by failure, and for persistency and tenacity to continue until the desired result has been accomplished.

In this work more than in any other mathematical discipline the student depends on the power of mind which he has acquired—not alone by the study of mathematical subjects, but also as a result of his whole intellectual training. But it takes a greater maturity of mind than is attained by the average high school pupil to derive these mental benefits from the study of geometry, and to acquire a thorough knowledge of geometrical concepts and facility in solving constructions. This remains true even if an undue amount of time and energy is expended, both by the pupil and teacher, on the study of the subject.

One who has never studied algebra, who has never heard of equations, and who has not even had a systematic training in arithmetic, may yet be able to solve some of the most difficult problems of algebra. But I doubt whether any one, however able, can by observation alone and without any systematic knowledge of the principles of geometry, arrive at some of the properties and proofs, of the triangle, say.

The steps of a geometrical proof should follow each other

in a strictly logical sequence. This, however, is not always the case. Many proofs require a number of auxiliary lines which to the beginner appear more or less artificial. In many others the successive steps are not always connected so as to enable the student to deduce each step from the preceding by purely logical reasoning and with the aid of principles previously derived. In reproducing such proofs the learner will depend also on his memory. It is an interesting question as to how much the memory has to do with the derivation of mathematical principles and laws. None of us can remember all the proofs we have studied, nor is it necessary that we should. An examination in geometry ought therefore to be conducted in such a manner as to convince the examiner that the student has obtained the mental benefit which the study of the subject is capable of yielding. Such an examination should consist of constructions and proofs of theorems commonly called original exercises. The teacher will soon learn from the manner in which the student attacks these, the strength of his geometrical development.

The growth of the mental powers, the development of the mind, is a matter of evolution. The degree of difficulty of an idea, the amount of meditation and reflection necessary to make it entirely clear to the student, must be adapted not only to his natural ability, but also to his age.

To derive the fullest mental benefit from the study of formal geometry and especially from practice in working constructions, the subject should be taken up in college. The study of geometry is, in my opinion, more difficult than the ordinary course in logic, which, as a rule, is reserved for the upper classes in college.

There seems to be no provision in either the secondary school or the college, for the student to get any idea of the possibility of a space of higher dimensions than the one known to us. The idea is extremely fascinating and is excellent material for meditation and reflection. A system in which we are led by logical deductions to the conclusion that a hollow sphere can be turned inside out without breaking its shell, in which there are five conics instead of four, and so on, is one well calculated to stimulate thought and meditation. If geometry were taught in college, where students are more mature such topics might be taken up. One of the purposes of education should be to give the

learner a thirst for knowledge and a desire for meditation and reflection. How much happiness is to be derived from the ability to meditate! The man who meditates does not feel lonesome, time is not heavy on his hands. He who enjoys meditation and reflection will not seek pleasure in diversions which he cannot afford, nor in those which undermine his health and maybe his morals.

Although geometry of all mathematical disciplines may best serve the purpose of education, yet on the whole there is not as much attention paid to the study of geometry as to the study of algebra. Not only is geometry the more difficult subject, but algebra appears to be the more practical. The solution of many problems is greatly facilitated by the use of the equation. By means of algebraical processes, interest tables, tables for discount, etc., have been computed.

On the other hand geometry has little practical application and its truths are of little interest in the ordinary pursuits of life. Its only purpose is to increase the power of mind of the learner, to develop in him the ability to draw logical inferences from accepted premises. If its study does not accomplish this purpose, it must be considered a failure, a waste of energy and time, not only retarding the development of character, but having a pernicious effect on it.

There is in general very little plane geometry used in the higher branches of mathematics, in calculus for example, or even in analytical geometry., In these subjects most operations are performed analytically. To this is due the greater importance which is attached to the study of algebra. I ought, however, to add that some of the problems of plane geometry which occur in engineering are of greater difficulty than the average pupil with but one year's instruction in the subject is able to solve. Problems relating to the division of plane figures, or such as arise in connection with the laying of curved railroad tracks, or of tracks leading to a turn-table, require a high development in geometrical methods and great facility in geometrical construction.

Geometry is generally acknowledged to be a strong educational medium. Its ability to develop power of mind is very seldom questioned. It is therefore the more surprising that a member of our own craft should give expression to the following:

"We had also fondly cherished the traditional belief that geometry is a tonic for all mental shortcomings (and we do not propose to retreat completely from this position) but we do not now dare to place as much confidence as heretofore in the theory that wits sharpened on geometry will therefore present a keen edge in all other phases of mental activity. We have seen so many good geometricians who seemed to possess small logical sense in other affairs and so many good logicians who have small knowledge and little facility in geometry, that we can no longer believe in the inevitable transference of power gained in the study of geometry to effective use in other lines." *

Casual intercourse with a person is not, as a rule, of such a nature as to enable us to judge of his mental powers. We are inclined to consider a person, who by constant practice is able to perform his duties without much effort, as possessing extraordinary ability.

The mind which is capable of deep thought and deep reflection does not always possess the superficial brightness often mistaken for mental power. The person who is able to speak of his special work most convincingly has not necessarily a logical mind.

It is a matter of my personal observation that mechanics and artisans who possess great skill, do not always have the power of mind to apply it in the most advantageous manner. It is one thing to build a house and another thing to furnish it. It is one thing to have skill and another thing to make that skill most efficient. Little as we know whether a person has been benefited by a certain study, we know still less whether he would have been more benefited by another.

There is at the present time much discussion about the new education, which I take to be utilitarian education. In spite of this tendency, the purpose of the school ought to be first and foremost, to develop the character and the mind of the learner. Unless the school can accomplish this purpose, it should not undertake other problems, however practical they may appear. The best inheritance we can leave our children is strong bodies and honest characters together with strong minds.

We cannot give our pupils directions for all the problems and conditions they may meet in life. Even in the professional

* THE MATHEMATICS TEACHER, July, 1909, p. 161.

schools we cannot prepare the student for all the difficulties he may meet in the practice of his profession. New conditions may arise which will leave him helpless, if he has not the power of mind to conquer and to solve them.

I am in this connection reminded of the following story: A little girl was instructed by her mother how to act at a party which she was to attend. The mother was careful to tell her what she should say when the cake was offered for the first time, and what she should say when it was passed the second time. At the party it happened that the cake was offered a third time. Not having been instructed what to say in this case, the girl recalled that her father so tempted had remarked, "Hell, I will have another piece."

A new education should be such as to prepare the student for the ever new and changing conditions that arise in the evolution of mankind. With every day our environment changes. Education to be adequate must prepare the student to keep pace with the progress of the world. That progress expresses itself in keener competition, in an enlarged field of activity, necessitating greater discrimination, greater resourcefulness, greater ability to exploit new enterprises, and to distribute the ever increasing products of manufacture and invention. Education should ever be directed towards increasing the power of mind of the individual to meet and cope with all kinds of activities in their ever-broadening aspects.

The inventions of the last decades disclose some of the wonderful possibilities of applying the forces of nature. The telephone, the flying machine, and the speed attained on land and sea are a few achievements of which we formerly never dreamt. We can only conjecture how much more is in store for those who have the necessary mental power. With all deference to the statement in the Bible that, "Thou shouldst gain thy bread by the sweat of thy brow" (Gen. 3, 19), I cannot help feeling that man was put on earth not merely to provide for all his needs by laboring from sunrise to sunset, but for the betterment of the world, for his own intellectual and spiritual improvement, for meditation and for proper enjoyment of life.

If we but discover the true character of the forces of nature, we may be able to supply our wants without the expenditure of much energy and time. I feel that the exploit-

ing of the natural forces by mechanical processes and devices is still in its infancy. I feel that forces of nature which will enable us to gain the comforts and necessities of life surround us in abundance. We must only learn to harness and apply them. What we need is keenness of perception, power of mind, power of discrimination and all of those qualities that education improves, if it does not create.

Education must give the learner that attitude of mind which distinguishes the educated person from the uneducated. It is not the number of facts a person knows, it is the attitude of his mind, and the manner in which he thinks and acts that distinguishes him from those who have not had the advantages of education.

Even a foreign language should be taught for the mental development which it is capable of affording. Unless the student devotes more time than is generally allowed in schools and colleges to the study of a foreign language, he will not acquire the knowledge necessary for its practical use. In fact, a foreign language can be acquired only by long and continued practice, or by living among those who speak it.

We cannot arrange the curriculum of the elementary school, the high school, or the college with a view to the needs of the few. We cannot teach mathematics in such manner as will enable the one girl out of ten millions to use it if she will become a surveyor or an engineer. All our teaching must be done with the single view to what is good for the majority—to what will inspire all the pupils and help all of them to true happiness and high ideals.

I feel that the mathematical subjects taught in the secondary schools, are, on the whole, selected and arranged with the view of satisfying the requirements of the colleges and of the engineering schools. Students of the latter must have a knowledge of all the elementary branches of mathematics if they are to follow the instruction in analytical geometry and calculus. But the average high school pupil is unable to thoroughly understand and assimilate all the mathematical subjects covered in his course in the time allotted to their study. He is as unable to acquire a thorough knowledge of geometry and to drive the mental benefit which it is capable of giving him, as the college student is unable to thoroughly understand, digest and assimilate

all the ideas in calculus as they are presented in some of the most widely used text-books, and to acquire facility in performing its operations. The teacher of mathematics, like the teacher of any other subject, must never lose sight of the fact that his subject is only a part of the curriculum, and that the pupil must divide his time between all the subjects of his course.

One cannot help viewing with apprehension the ever increasing size of the text-books in mathematics. This tends to foster the pernicious habit of superficiality and acts against all the true purposes of education. We mathematicians are not the only sinners in this regard. While I am writing this a relative of mine is studying from a text-book on a medical subject containing about 1,000 pages. I understand that three hours a week for two terms of one half year each is given to the subject. I feel that it would be quite impossible for a person of average ability occupied with the many other subjects of the modern medical school curriculum, to acquire even a superficial knowledge of such a work in the time allowed to it. It would serve the cause of education if the text-books in most subjects were greatly reduced in size and if the student were held to a thorough knowledge of each and every idea presented in the book.

In formulating a course of study in any subject the principal questions ought to be, how much ground can be covered by the average student in a thorough manner so that he may gain a perfectly clear idea of each and every point in the subject in the time given to it.

Even the more difficult parts of elementary algebra, like the reduction of fractions and the reduction of surds, now given in secondary schools, should be taken up in college where the student is more mature. I know how difficult it is to break with established traditions. It will take the united efforts of all of us to bring about a change from existing conditions, to reduce the amount of each of the mathematical branches taught in secondary schools and in colleges, and to arrange the curriculum so that at any point the student may be properly prepared and sufficiently mature to successfully take up the next subject.

The present tendency seems to be towards higher entrance requirements, which as I understand it, implies more ground to be

covered. This can in general only be accomplished at the expense of thoroughness.

We should increase the intensity and efficiency of a student's knowledge of mathematics, rather than extend the scope and quantity of the subject matter covered. One clear idea, one proof which the student thoroughly understands, in which he knows the reason for every step taken, and which is entirely his own mental possession, is of infinitely greater benefit to his mental development than if he were to go over a great many theorems in a perfunctory manner. Instead of developing thoroughness, patience, perseverance, independence in thought and action, such a course fosters those pernicious qualities which it is the duty of the teacher to correct and overcome.

In the rush of our daily work, and in the rush of all our activities, we are apt to forget the true purpose of education. If the foundation of a subject is thorough, not only will this be fulfilled but also the pupil will be equipped to continue the study of that subject.

I cannot help but think that we send children to school much too early, that a great deal of the work in mathematics which the average pupil does in the elementary school and in the high school is beyond his mental abilities. We have over and over again heard of children who did not commence to study until much beyond the age at which the average child begins to attend school, and who have nevertheless accomplished more than those children who have been confined in the school room from their very early childhood. We all know how hazy our ideas were on some of the subjects we studied at school. It was not always due to lack of application on our part, more often it was the immaturity of our minds and the state of our mental faculties, even though they were normal for our age. No idea should be submitted to a child until his age and mental development permit him to gain a perfectly clear understanding of the idea and to assimilate and retain it.

Four courses in geometry should be given. First, a preliminary or introductory course. In this the pupil should draw simple figures, including such as may be termed geometrical ornaments. Squares, hexagons and triangles are laid out so as to form a compact plane—a plane is paved, so to speak, with these figures. Those children who have had the advantage of

Froebel's ideas as given in properly conducted kindergartens have in the study of geometry a decided advantage over those who have not had such instruction. Froebel believed in the early training in geometrical *forms*.

In a second course, called observational, inventional, or intuitive geometry, the pupil's conceptions of space-forms are extended; the properties of some of the geometrical figures are deduced by mental and physical inspection, that is by observation and by actual measurment. In this course the student will accumulate geometrical ideas and extend his geometrical instinct. The training derived from such a course is very important for the development of certain faculties in the young, especially those of observing, of discovering, of going open-eyed through the world. These faculties should be trained early. A desirable faculty in a person whatsoever his occupation may be, is the ability to estimate with some degree of accuracy, lengths and distances. This should be developed in the course in inventional geometry. Lengths of objects and of lines running in different directions are estimated by the pupil, and then by actual measurement he convinces himself of the accuracy of his estimate.

A most useful exercise which ought also to be taken up in this course is drawing to scale. This is one of the very few applications which geometry has in practical life. All of us might have occasion to find the distance between two places from their relative positions on a map, or to determine the dimensions of a building from its plan.

In a third course—demonstrative or formal geometry—the fundamental conceptions, formal proofs of some of the theorems and some easier constructions should be included. In this course the pupil deduces the properties of figures from axioms and definitions by a sequence of logical reasoning. He verifies by proof the truth of some of the properties he has previously obtained by observation and actual measurement. In observational and in inventional geometry a correct and accurate figure is indispensable. On the other hand for the formal demonstration of a theorem a correct figure is not essential. In fact, geometry has been defined as the science of "correct reasoning and bad figures."

A fourth course comprises the proofs of the more difficult theorems and the solution of geometrical constructions. This course should be taken up when the student is more mature, in the higher classes of the college.

It is a question whether inventional geometry should precede or accompany the course in demonstrative geometry. That is, whether intuitive geometry should be taken up first and then demonstrative geometry, or whether a pupil should be made to discover by measurement, etc., the properties of a figure and then by proof verify his estimate. With the latter mode of procedure he will better understand what he is expected to prove and perhaps be better prepared to proceed with the demonstration. This method is to some extent used in algebra. From a particular example a principle is deduced, and then the principle is formally proved. In this, inductive and deductive methods go hand in hand.

There is and has been a great deal of dissatisfaction with the teaching of geometry. This dissatisfaction is concerned, however, more with the inadequacy of the system of Euclid and of some of the systems designed to replace the Euclidean, than with the teaching of the subject as it actually effects the student. It is to be hoped, however, that an improvement in the system will bring about an improvement in the student's knowledge of the subject.

The principal objections to Euclid lie in the unsatisfactory definitions, axioms and proofs. Yet the definitions and axioms which have been suggested to replace those of Euclid have also been found wanting.

A definition should enable us to construct a geometrical concept in a clear and unambiguous manner by the aid of such concepts only as have already been defined.

It should be possible to completely construct a geometrical concept from its definition without the aid of imagination or of an *a priori* knowledge of it. When I say without imagination I mean that the latter should not supplement any defective parts of the definition. It is a question whether it is possible to include in a definition all the properties of a geometrical concept so that it will not be necessary to supplement it with any *a priori* knowledge. Euclid defines a point as that of which no part can be taken. But there is

a sense in which it may be said that no part of a molecule or of an atom can be taken. Again a point is often defined as the extremity of a straight line. This definition tells us what a point is in connection with a straight line, but not what it is independent of any other geometrical concept. Euclid gives the following definition of a straight line: Any line which lies evenly between the points in itself is a straight line. The definition of a straight line as the shortest distance between two points requires the definition of distance. The definition of a distance is based in turn on the definition of a straight line. A straight line is also defined as an extension in one dimension, but this requires first the definition of dimension. A straight line is sometimes defined as the path of a point moving constantly in the same direction, but direction cannot be defined without the idea of a straight line. We say that two straight lines have the same direction if they both pass through the same infinitely distant point, but an infinitely distant point cannot be defined without the conception of a straight line. No satisfactory definition of a straight line has yet been given.

I think it was Tyndall who said that no one ever gained a clear notion of what a straight line is from its definition. We are born with the intuitive or preconceived conceptions of a point and a straight line, which we call *a priori* conceptions.

Neither has a satisfactory definition of space been given. In fact, it is impossible not only to define space, but even to describe it.

Euclid defines a plane as a surface which lies evenly between the straight lines in itself. Besides not being clear, this presupposes the definition of a surface, which is difficult to define. The same objection holds against the definition: A plane is a surface such, that if any two points in it are joined by a straight line, the line will lie wholly in the surface.

Another definition of a plane is: A plane is obtained if a point outside of a straight line is joined to every point on the line. This definition does not determine all the points on the plane. It does not include those points which lie on the line passing through the given point and parallel to the given line. But parallel lines cannot be defined without first defining a plane. Still another definition of a plane is: A plane is the locus of all

points which are equally distant from two fixed points. This definition makes the assumption of a third dimension necessary and also the conception of equal distances. The definition: "If a right angle is revolving about one of its sides, the second side describes a plane," has the same difficulty.

The Italian mathematician Peano has given a definition of a plane which is rather too difficult for the beginner. It is: If one of three given points is connected by a straight line to a point on the line joining the other two points, and if from any point in this connecting line straight lines are drawn to points on the straight lines joining the three given points, we obtain a plane.

Euclid defines a plane angle as the inclination of lines one to another; that is, of two lines in a plane, meeting each other and not lying in a straight line. This definition would exclude a straight angle or a zero angle. An angle is also defined as the difference in direction between two straight lines intersecting in a point. But direction is a qualitative and not a quantitative conception. It therefore cannot be increased or diminished. Direction ought also be taken with respect to a fixed line.

Euclid's definition of a right angle is an angle which is equal to its adjacent angle. This requires the definition of adjacent angles. It also requires the knowledge that when two adjacent angles are equal, each is a right angle.

Another geometrical concept which is difficult to define is symmetry. Nothing will make this as clear as a concrete illustration. A pair of gloves which are equal in all their parts, yet cannot be made to cover each other, since the parts are arranged in opposite directions, are symmetrical.

The question arises, how many definitions are necessary to build up a consistent geometry so that no idea shall be assumed without having previously been defined. Most text-books fail to define for instance the bisector of an angle, or when three points are on the same straight line, etc. Three points are on the same straight line when the distance between two of them is equal to the sum or the difference of their distances from the third, etc. We have as much difficulty in properly defining some geometrical concepts as physicists have in defining matter and force, psychologists, mind and perception, and economists,

money and value. We acquire a knowledge of these ideas by repeatedly using them.

It is an important question whether we can evolve any concept of space-forms from abstract descriptions. No definition or description of a cube will convey to the immature child as definite an idea of a cube as will the seeing of it. There is also the question whether the theorems of geometry are constructed from fundamental concepts only, or also by the aid of experience and observation. It would be rather difficult to deduce the theorem that the sum of the three angles of a triangle is equal to two right angles from the definitions of an angle and a triangle only. It is a question whether we can obtain the theorem, that a straight line cuts a circle in two points, from the definitions of straight line and circle, without the aid of observation. It is an interesting inquiry whether we ever follow in geometry pure reasoning, entirely separate from our experience.

A more serious defect in Euclid is the inadequacy of the axioms. The 11th axiom: "If two lines are cut by a third, and the sum of the interior angles on the same side of the cutting line is less than two right angles, the lines will meet on that side when sufficiently produced," has defied the ingenuity of the ablest mathematicians to make it dependent upon simpler truths, that is, to prove it. It is quite surprising to find that in Euclid this 11th axiom is the converse of the seventeenth proposition: In a triangle the sum of two angles is less than two right angles. There are a number of substitutes for the eleventh axiom. One of them is: Through a point only one straight line can be drawn parallel to a given straight line, which sounds as if it were in no less need of proof.

Lobatschevsky and Bolyai, assuming that two parallel straight lines do meet, or what is the same, that the sum of the angles of a triangle is less than two right angles, thus eliminating the eleventh axiom of Euclid, have constructed an entirely consistent geometry called non-Euclidean geometry.

It cannot be denied that non-Euclidean geometry would serve much better that Euclidean geometry to develop power of mind in the learner, and the faculty of adhering to a sequence of strict logical reasoning. Since the truths of the theorems of non-Euclidean geometry do not agree with our

experience of space, their proofs require much closer concentration of mind. In Euclidean geometry our reasoning is supported by our experience of the space about us, while the results of non-Euclidean geometry contradict our observation and experience.

Some of the ablest mathematicians, among them Poincaré, Hilbert, Peano, Pasch, and Worpitzky have devised systems of axioms. But no system which is consistent, necessary, and sufficient has yet been established. It is amazing how greatly mathematicians differ as to the number and kind of assumptions which they consider necessary to build up a consistent and complete geometry. In this connection the question arises whether there is a finite number of axioms, or whether the number of axioms is infinite, and what is the first, the simplest axiom? Veronese says that he should like to be shown how a complete and consistent geometry can be constructed from Poincaré's axioms.

Mathematics is in more than one way receiving credit to which it is not entitled. Our science is looked upon by the representatives of other sciences as a perfect one, as the one in which definitions are complete, clear and unambiguous. It is therefore the more to be regretted that as far as the fundamental principles of our science is concerned, it may never be satisfactorily settled.

Several attempts have been made to devise a system of geometry different from the one which has come to us from Euclid. Some writers have tried to make the axioms more complete. Others have included some of the concepts of modern geometry. Others again have tried to evade the parallel axiom by first proving that the sum of the three angles of a triangle is equal to two right angles. One of the best known of the latter efforts is that of Thibaut. The proof is correct, but it assumes the revolving of a straight line about a point through 360 degrees to be equivalent to the revolving of the line about three points making the sum of the revolutions equal to 360 degrees.

The methods of proof in Euclid show serious defects. In some of the Euclidean proofs it is often difficult to recognize the connection between the beginning of the proof and what is to be proved. Euclidean proofs lack homogeneity of sys-

tem, and a general method of procedure. The proofs of a set of theorems even when they are closely related, have as a rule very little in common. In Euclidean geometry a proof refers to a fixed position of the straight lines and points of a figure. As soon as they change their positions the proof changes too, and very often the entire method of proof. Yet they all impress us with their high degree of excellence and ingenuity.

Although many of the writers on the subject of geometry are full of praise for the advantages to be derived in applying the principles and ideas of modern geometry—as discovered by Poncelet, Moebius, von Staudt and Steiner—to the study of elementary plane geometry, I have so far failed to discern any of them. Euclid once told his King, Ptolomæus, who found the study of geometry difficult, that "there is no royal road for geometry" but the followers of the modern geometry seem to think that this royal road has been found. We hear and read a great deal about the principle of duality, which it is claimed unifies the theorems and proofs of geometry and brings system into its chaos. This principle has greatly facilitated the study of curves and surfaces, of theorems and problems referring to groups of circles (*e. g.*, Malfatti's problem and the problems of Apollonius), and to inscribed and circumscribed polygons. I have failed however to find that the modern ideas possess any advantage in the study of the elementary properties of the triangle and the circle and in the proofs of those theorems which form the foundation of a large number of constructions. In modern geometry we have a much larger number of ideas to master than is necessary to follow Euclid. Concepts like infinitely distant points, infinitely distant lines, which play such an important part in modern geometry, are difficult ideas for the beginner. These concepts are necessary for the higher branches, but should be omitted in an elementary course in plane geometry.

Most of us have studied these ideas when we were of mature age and with a knowledge of Euclid. It is hard to abstract from the knowledge which we possess. I personally doubt that such an exclusion of ideas is at all possible. The beginner with no knowledge of Euclidean geometry would, I think, find great difficulty in following some of the ideas of the modern geometry.

I would hesitate to use with beginners the text books on the subject, at least those which I have seen.

It is quite significant that those who advocate the new ideas of geometry use Euclidean methods in solving constructions. Some of the proofs in modern geometry are easier than Euclid's but they lack the vigor of the latter. There are, however, a few ideas in modern geometry which might be introduced to advantage in the study of Euclid. One is the idea of reciprocity, duality and inversion.

Two straight lines determine a point.

Two points determine a straight line.

Three straight lines in general do not pass through the same point.

Three points in general do not lie on the same straight line.

Through two points an infinitely large number of circles can be passed.

Two straight lines are tangent to an infinitely large number of circles.

n points of a plane, give $\frac{n(n-1)}{2}$ connecting straight lines, provided not more than two of them are on the same straight line.

n straight lines in a plane intersect in $\frac{n(n-1)}{2}$ points, provided not more than two of them pass through the same point.

Some German mathematicians who wish to see the teaching of geometry reformed are not satisfied merely to add to Euclid some of the theorems of the new geometry, as for instance, those for harmonic points and harmonic divisions, points of similarity, pole and polar. They contend that it is not the amount of material, but it is the method which characterizes the new geometry and which is distinct from that of Euclid. They wish the subject developed entirely along the modern lines.

In teaching geometry it must be carefully decided which of the multitude of theorems shall be proved. The proof of theorems like, "Every circle has one center only" and "All right angles are equal," etc., may be of greater mental benefit to the student, than the proof of theorems the truth of which is not self-evident. The problem is then to select those theorems whose

proofs will be of most benefit to the learner. A proof must be free from inconsistencies and from such illusory statements as would stamp it in ordinary language as a sham proof. Some theorems can be proved in several ways. It is important to select the one which offers the best training for the mind. There is good mathematics and bad mathematics, if such an expression be permissible. We may prove a theorem or solve a problem in a short and direct way, or in a long and tedious way.

Another important question is whether a proof should be based as much as possible on first principles or on a succession of propositions. The Pythagorean theorem, for instance, can be proved in many ways, depending on the number and kind of basic theorems which are used.

Whatever system is used, whatever method is adopted in the teaching of a subject, no matter how excellent the text-book, it is after all the teacher, his ability to impart knowledge, to instill in the young the love for knowledge and for truth, his personality and his character, which is the most decisive factor in any system of education.

The knowledge of a teacher must be thorough and broad if he is to be able to meet all the possible conditions which may arise in the teaching of his subject. A teacher of geometry should be acquainted with all its systems and all its methods of construction, with Steiner's methods of carrying out a geometrical construction by means of a straight edge and a fixed circle in the plane without the use of compasses. He should be familiar with Mascheroni's methods for performing constructions by means of compasses only and without a straight edge, a straight line being given by means of two points only. No fixed method for the teacher to follow can be given. Whether to use the heuristic method, or the inductive or deductive, or the genetic method, will depend on the average mental capacity of the students, and on the special conditions and circumstances which may influence their minds. The method suitable to a bright spring day might prove ill adapted to a cloudy, heavy, fall day. Here is where the ingenuity and the ability of the teacher make themselves felt.

The student must thoroughly understand the content of a theorem, the meaning of each condition, and what is required to be proved. It is a pernicious habit, and the teacher must

carefully guard the student from giving way to it, to read over a problem or a theorem, and then before thoroughly understanding its content and meaning, to attempt its solution or demonstration. To make sure that the student understands what he is to do, he must be made to repeat the theorem in a wording different from that of the book. If a person is unable to express a thought, it is as a rule an unmistakable sign that the thought is not entirely clear to him. A student's claim that he understands the idea but is not able to express it, is very often unfounded. If he does understand the idea he will find words to express it, just as the teacher who has entire command of his subject will always be able to impart it to the learner. My experience has been that most students have difficulty in understanding quantitative conditions. In a theorem like the following: "In a triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides," the learner should understand that if the angle A of the triangle ABC is bisected by a line AD, the point D divides the side BC into two parts such that if AB is $\frac{3}{4}$ of AC, then BD will be $\frac{3}{4}$ of DC.

Few students thoroughly understand the meaning and significance of theorems like—"A line drawn parallel to a side of a triangle divides the other sides proportionally."

The young find thinking a hard task. A pupil will undergo the labor of memorizing a proposition and its proof rather than follow the more difficult course of concentrating his mind and trying to understand the meaning of the theorem and the successive steps of its demonstration. If the teacher is uninquisitive and is satisfied with the work as put on the blackboard or paper by the student, the latter is fairly sure of a high mark on a memorized proof. It is the duty of the teacher to require of the student the reasons for every step and to convince himself whether he thoroughly understands each condition in the theorem and each point of the proof. The teacher should not take any knowledge of the student for granted. What appears to be self evident to us may be entirely obscure to the student. He may not even be conscious that the idea is not clear to him until he is questioned about it. A good plan is to prove a theorem from the given conditions, and then after accepting its truth, invert the order of reasoning and see whether the given conditions must

prevail. The student will then better understand the reasons of each step of the proof. He will also better appreciate the necessity for all the steps taken in the course of the proof.

The teacher ought to work before the class a number of constructions, showing in detail the methods by which their solutions are obtained. In this way each student will acquire a part of the teacher's power of mind and of his knowledge and ability. Unconsciously, he will imitate the methods of the teacher. It is therefore extremely important that the pupil be instructed by a person who not only possesses a thorough knowledge of his subject, but who is equipped as well with the best methods. But if a student is to acquire the faculty of independent thought and action in striking out for himself, if he is to get the mental strength necessary to overcome obstacles different from any he has ever met before, and if he is to adapt himself to new and changed conditions, he will have to perform such constructions for himself. To find a solution will not only give him power of mind but will give him the joy and satisfaction of accomplishing something by his own strength. Searching for a solution will give the student a better grasp on the principles of the subject.

The explanations given by the teacher in any subject, and especially in mathematics, must be explicit and full. In the course of the discussion of a topic, the most elementary idea, the easiest formula with which the student is supposed to be familiar and which is involved in the explanation, should be carefully restated and expounded if by so doing the student's understanding will be clarified. Even if the student is familiar with the ideas used in the explanation, it may take some time for him to recall them, unless his mastery of the ideas is such as come, only from constant practice and repeated application. A good plan is for the teacher to restate all the principles and ideas involved in a proof or in the solution of a problem before proceeding with the work.

In building a house we do not erect it to the first story window and then order the window frames, thus interrupting the work. We order all the parts necessary for the entire structure before we commence building and then use each part as we need it. We must never forget that we teachers acquire by constant repetition a high degree of proficiency in the ideas

associated with our work. We have our definitions, principles and formulæ, so to speak, at our fingers' ends, but the student, no matter how able, can use them only with more or less effort.

It is highly important that in the course of an explanation no gaps should occur in the student's understanding. If he is to follow the explanations of the teacher and at the same time supplement them with such ideas as the teacher uses without restating, the student has two functions to perform, where one would be a sufficient tax on his mind.

While the teacher must insist firmly and uncompromisingly that the student do his full duty, he must never fail to encourage and stimulate the earnest worker with a word of praise, nor to reprimand, punish, or criticise the student who shirks his work. There is no one of us who is not encouraged and stimulated by sincere and well-earned praise. There is something wrong with us when we no longer care for the opinion of our fellow men.

If the student is neglectful of his duties, it is not sufficient that the teacher reprimand him and then dismiss him from consideration, nor is it sufficient in addition, to report him to the proper authorities, or to his parents or guardian, as deficient. It is the duty of the teacher to keep after the student and by convincing arguments get him to do his duty.

In urging students to do their duty and in rebuking them, great tact must be exercised and great care must be taken that the pressure brought to bear shall not have an effect opposite to the one desired. The student, who on account of inability or poor health, or for any other reasonable cause, is prevented from doing his work, must not be rebuked. He needs all our help and sympathy, all our kindness and encouragement.

There is no doubt that our first impressions are the most lasting. We remember incidents of childhood and youth much longer than those of manhood. Because of the novelty of the experience, they arouse the curiosity and produce a stronger impression on the mind. It is therefore of the greatest importance that those of us who are entrusted with the education of the younger pupils should possess the highest qualifications. If a difference in quality in the teachers for different grades be permissible, or if it be impossible to procure for all grades and classes teachers of the highest attainments,

the younger pupils should have the teacher who possesses the highest qualities of character and mind and the strongest personality. What is true of the lower grades of the elementary school is also true of the lowest classes in the college.

We are just passing through an era wherein young men rule in most of the occupations and professions. While we see unmistakable signs that this era is passing, yet there are positions and occupations in which the young man with all the push and physical vigor of youth is preferable to the man more calm and less enduring. But the calm repose of character and mind, and the ability to apply them for the benefit of the young, come only with age. There are many occupations in which one must serve as an apprentice before reaching the position of independent master. Those who intend to teach should be made to serve as apprentice in the class room of an experienced teacher. The strictest supervision by principals and superintendents cannot take the place of this suggested course. There are many young men and women, who after graduating from school and college while preparing for one of the professions or more remunerative occupations, are taking positions as teachers. They first teach then they go out into the world to gather experience. But the reverse procedure should be one of the requirements of the teaching profession. Those who intend to teach should by experience and observation broaden their minds, and then begin teaching. They will then understand human nature better. They will know what is expected of the future citizen and be able to influence him towards it. The teacher who uses his position as a stepping stone to something more remunerative is in most cases doing an infinite harm to the pupils entrusted to his instruction. Not considering teaching as his life work, he will as a rule not concentrate his efforts on better fitting himself for his duties.

UNIVERSITY OF PENNSYLVANIA,
December, 1909.

MATHEMATICS FOR SERVICE.*

By Ernest R. von Nardroff.

For a long time mathematics has enjoyed the position of first place in the elementary schools, and in the high schools the time devoted to the subject is nearly as great as that for any other subject. Almost daily for ten years, reaching from the first primary grade up to the conclusion of the second year in the high school, a pupil is required to bend his best energies toward the mastery of mathematics. What are the results? It is needless to state that the average graduate going into business or into any of the learned professions, except engineering, fails to put his mathematics to any use whatsoever. Beyond the requirements of keeping one's accounts and calculating interest and taxes, the ordinary citizen has no need of formal mathematics. The result is that practically everything in this subject learned in the high school is promptly forgotten.

How does it stand with the student of engineering? After taking all the mathematics he can get in the high school he pursues the subject further, either in the pure or applied form, for four years at college. Probably no other subject in an engineering school causes as many failures, and yet I feel confident that not one engineer in ten avails himself of any of the higher mathematics that caused him so much pain and trouble. Nearly everything he does of a mathematical character is done with the aid of formulæ, or by well-settled rules. Anything like an independent mathematical excursion, for which he is miserably prepared, is generally quite out of the question.

Let us finally consider the student of physics. I mean one who aspires to be master of the science, both experimentally and mathematically. If, after his graduation from college, he attempts to take up the reading of advanced mathematical physics, such as the works of Maxwell and Rayleigh as well as many of the articles found in periodicals like the *Philosophical Magazine,* he finds himself baffled by mathematical material

*Read at the meeting of the New York Section.

quite strange to him. It becomes necessary for him to interrupt and retard his studies in mathematical physics by long detours into the field of pure mathematics. The reason for this is, not that he has not studied a sufficient *quantity* of mathematics, but that he has not studied sufficient *kinds* of mathematics. He needs a little from each of a great many mathematical subjects. In his high school and college courses his time was taken up studying in only a few subjects a mass of material that has no bearing whatsoever upon his work.

If, then, class-room mathematics has thus signally failed to render service in the world of realities, some other defence for retaining it to the present extent in our curriculum must be found. There are so many other subjects of a modern character pushing to the front and claiming for themselves the highest service, that, unless mathematics can claim for itself as much as is offered by these other subjects, it will surely be forced to give way. It can, indeed, make a very strong claim, a claim for *culture*. No one who has witnessed a class of eager pupils, vying with each other to be first in finding a solution to some geometrical problem can doubt for a moment the tremendous value of the culture derived from such exercise; and I am glad to say that in recent years the teaching of geometry has been very much improved in the direction of enhancing the culture obtainable from its study. But here, again, mathematics must compete with those modern subjects that also claim high cultural value, and so, in spite of all that can be said and done, mathematics will presently be pushed very hard.

This condition of affairs has been well recognized by representative mathematics teachers, and so several plans have been suggested for making mathematics of value for *service* as well as for culture. "Service" is here employed, not in the indefinite general sense that might include mental growth, but in the sense of direct utility.

One plan that has been suggested is to introduce a large number of technical problems drawn from the trades and allied sources—problems in sheet metal work, brick laying, building construction, railroading, etc. I fear that this plan will not lead to the desired result because to boys and girls not interested in the trades represented, the most important problems would be exceedingly dull. Besides in many cases very lengthy

explanations would be required unless the plan happened to be worked out in a school provided with the corresponding shops.

A second plan is to introduce illustrative problems from the field of mechanics and physics, as, for example, problems relating to momentum, kinetic energy, specific gravity, and calorimetry. I cannot approve of this plan because I feel confident that it is pedagogically wrong to illustrate a principle by means of strange material that is harder to understand than the principle being illustrated. It must be remembered that in our high schools physics does not come until the third year.

A third plan, already fairly well worked out, is, though limited in application, somewhat more promising. I refer to the introduction of problems from strictly every-day life. But I should like here to caution against the introduction of methods that are foreign to those appropriate to these problems. For example, in calculating the amount of carpet required to cover a floor one often meets with methods that would lead to results very different from those obtainable from a reputable carpet dealer. The same is true for paper hanging. The proper methods should be learned from practical men. However, the change here suggested, although an important one, is so simple as not to require further comment. Some additional problem material may be obtained by looking into the out-of-school activities and interests of boys and girls.

A fourth suggestion, which I myself would make were the time ripe for it, is to begin physics in the lowest grades. There can be no question of the fact that physics is by far the most comprehensive field for the application of mathematics. Scarcely any subject of mathematics has failed to find some application here, and, further, in many cases the demands of physics have been so far beyond the powers of mathematics to supply, that mathematicians have been either forced then and there to enlarge their own domain or have had it done for them. Should physics, some time in the future, obtain a place in our curriculum as conspicuous as that which mathematics now holds, there would be no lack of material for illustrating the highest mathematics, and there could be no question of the service that mathematics could and must per-

form in return. But this is a condition not likely to arise in the very near future.

I am, therefore, forced to offer now, for present carrying out, a fifth sugestion. This suggestion is to teach additional mathematical subjects, and in all subjects to emphasize those processes that are useful in scientific and practical fields as well as in every-day life. The illustrative problems should relate to perfectly familiar matters as practical as possible. It will be very desirable in high schools to go far beyond the subjects ordinarily taught. In practical fields a considerable demand is made upon analytical geometry, and calculus is a tool for constant service. It will only be possible to study these subjects by gaining time through the elimination from our present course of all material of little or no practical value, direct or indirect. This elimination is of the greatest importance because, besides the need for time in taking up higher subjects, it is necessary to gain time for more thorough drilling upon such phases of our present subjects as are of first importance in practical work. It will also be necessary to gain time by confining attention to simple problems and forms.

There will undoubtedly be some difference in opinion as to just what topics should be emphasized; which should be given only a small amount of time in passing; and which should be cut out entirely; and indeed I think that the first step taken toward carrying out the plan that I am suggesting should be a thorough investigation into what pure mathematics is really useful in higher mathematical physics, in navigation, in astronomy, in engineering, in the trades, and in ordinary life. In the meantime, for the sake of definiteness, it will not be amiss for me to suggest a few topics that I should like to see emphasized, and a few topics that I should like to see simplified or even eliminated.

In algebra, I would *emphasize* the solution of simple simultaneous equations by the method of substitution. The coefficients employed in the algebraic expressions should not always be simple ones, but occasionally should imitate practical conditions by the introduction of one or two decimal places. The pupils should become skillful in the solution of quadratic equations involving one unknown quantity, either by the use of a formula or by the method of completing the square. The

binomial theorem should be much emphasized, especially in the case of simple fractional powers. In this case the first few terms of the infinite series obtained will frequently serve as a suitable approximation in a complicated problem. The graphic solution of the higher quantics and of transcendental equations should be emphasized. It will not only leave the pupil in a position to solve with ease important problems that formerly required a deep study of the theory of equations, but will beside add very much to the cultural value of his study by giving him a distinctly new viewpoint. Further, graphic methods, when once mastered, may easily be remembered for a lifetime.

Much more should be made of methods of approximation, as for example:

$$1.0004 \times 0.9997 \times 1.0002 \div 1.0001 = 1.0002,$$

in which the principle involved is of frequent occurrence in physics.

Another method of approximation is one that I would suggest as a simplification of one commonly employed. The common expression, now current in scientific literature, for two hundred and forty thousand is 24×10^4. I would suggest that we employ the expression 10_4. The space gained in the following example is obvious, as is also the opportunity to perform it mentally with ease.

$$\frac{(2_3 \times 3_3)^2}{2_5} = 18_7.$$

Numerical solutions of algebraic expressions in which numbers are substituted for letters are of very great value. Another point to be emphasized is to solve an equation for each one of its letters in terms of the remaining letters.

In physics we frequently consider a quantity as being in *direct* or *inverse proportion* to some other quantity; for example in the expression,

$$a = \frac{c^2 q}{p},$$

it is important for the pupil to understand that a is directly proportional to c^2 and inversely proportional to d; and conversely, if the pupil is given the facts as to the relationship of

a to c and d he should be taught to construct the formula employing b as a constant. This is the kind of problem that is constantly arising in elementary physics. We discover experimentally certain relationships between the members of a complex set of quantities, and then seek to express these relationships by a single formula.

Incidentally, I would like to suggest the following method of collecting the principal facts in proportion. It occurred to me in connection with the diagram for Wheatstone's bridge. I express the proportion,

$$\frac{a}{b} = \frac{c}{d},$$

thus:

$$\frac{a}{c}\,\frac{b}{d}.$$

The lines, either horizontal or vertical, may be conceived to represent division when passed over mentally. The combination of the vertical and horizontal lines viewed diagonally may be conceived to represent multiplication. If then the two diagonal products are equal, it may be proved that we have a proportion,

(1) When we start in either of two ways with any one of the four letters;
(2) When all the letters are squared;
(3) When the rows are multiplied by the same or different amounts; and
(4) When either row is increased or diminished by any multiple of the other row.

I should also emphasize in connection with the study of algebra the use of logarithms and of the slide rule.

In algebra I would *omit* all factoring with the exception of that of the difference of two squares, and that for a perfect trinomial square. I do not recall any practical use of factoring demanding more than this. Of course Least Common Multiple, Greatest Common Divisor, and the addition of fractions having large polynomials for their denominators could not well be taken up without more factoring than I have indicated, but the processes just indicated are of little or no practical value when they happen to involve elaborate factoring. I ought here

to remind you that there is an immense difference between the appearance of a problem designed by a mathematics teacher and that of one arising in every-day life. The coefficients of real problems are rarely simple numbers. They usually have several decimal places and *appear* to be incommensurable. With such numbers many of the methods of factoring ordinarily taught break down completely. I therefore regard the teaching of such trial methods as very pernicious because, while the pupil may very easily solve seemingly difficult problems by them, he is acquiring a false sense of power. What applies to factoring itself applies to all that depends upon factoring.

I would eliminate the study of the square root of large polynomials; of the solution of simultaneous quadratic equations, except by graphic methods; and continued fractions, the practical use of which must be very limited indeed.

I believe much of the difficulty in the study of radicals can be avoided by the omission of the radical sign. Fractional exponents are simpler and accomplish all that is needed.

In geometry there are some propositions that are used thousands of times as frequently as others. I will mention a few of these that occur to me at the moment. They are those that relate to equality of triangles, to the equivalence of triangles, to the similarity of triangles, to parallel lines, to the square of the hypothenuse, to the angle at the circumference, to the relative areas of similar surfaces whether in two or three dimensions, and to relative volumes of similar solids.

Mensuration should also be emphasized, and in taking up this subject the pupils should be required to calculate from their own measurements the areas of plane diagrams; and similarly with the surfaces and volumes of various solids.

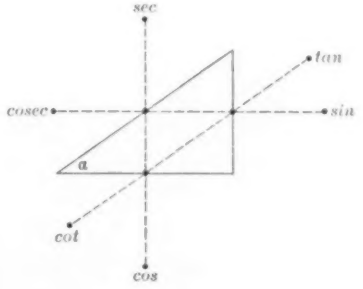
The solution of geometrical problems should not be confined to the use of the compasses and straight edge. The drawing board, with its T square and triangles, should be used also, for these are the instruments employed by those who are solving, in practical fields, the greatest number and variety of geometrical problems.

In geometry I would omit about one third of the second book, and one half of each of the third, fourth and fifth books.

In trigonometry, I would emphasize the solution of right angled and oblique triangles, the summation and difference

formulæ for sine and cosine, the relation between the sine and cosine, the use of a full table of trigonometrical formulæ, and the use of a table of the natural and logarithmic values of trigonometric functions.

One of the most valuable devices that I have ever met with for holding in mind trigonometrical relations is the following diagram.



The functions refer to the angle a. Assuming the dots to stand for the functions against which they are placed, or for the lines of the right angled triangle that bear them, the following simple statement will be found to cover an immense system of formulæ. *Any dot is equal to the product of its two adjacent dots.* These dots must either be on the same straight line, or else upon an imaginary circumference passing through the functions. As a corollary, any extreme dot is equal to the reciprocal of its opposite extreme dot on the same broken line. Thus:

$$\tan a = \sin a \sec a,$$

$$\tan a = \frac{\sin a}{\cos a},$$

and

$$\tan a = \frac{1}{\cot a}.$$

Or again, the base equals the hypothenuse multiplied by the cosine of the basal angle. I have found this diagram of the greatest use in the higher study of the refraction of light.

I would urge the study of surveying by the use of diagrams of fields, rivers, and mountain peaks. The pupil may select on the diagram appropriate points for taking sights; he may

use his ruler in places that in practice would require the steel tape, and a protractor where a transit would ordinarily be used; all to be followed by calculation of results from the data obtained.

Before passing to higher mathematics, I would like to invite attention to the following points in arithmetic: The use of tables of squares, cubes, roots and reciprocals; and a discussion of significant figures and of the customary methods of abbreviated multiplication, division and square root.

In analytical geometry only the simpler theorems relating to conics need be studied, but particular stress should be placed upon the study of the ellipsoid. Much importance should be attached to the study of polar coördinates in three dimensions.

In calculus I would first teach the process of differentiation, illustrating it in a few cases only. I should then provide the pupil with an elaborate reference table of differentials. With integrals I would proceed in the same way. I would endeavor to have the pupil appreciate the significance of differentiation through a contemplation of the variation in purely numerical quantities for which the letters of his equation may be conceived to stand; but I should also make much of the understanding of differentiation in connection with geometrical concepts, with thermal concepts, with pressure concepts, with motion concepts, etc. Also with any physical concept the pupil should understand what is meant by differentiation, not merely in terms of the physical concept itself, but also in terms of its geometrical or algebraic representation. Much is to be made of maxima and minima, a few singularities, involutes and evolutes, definite areas and volumes, and integration between limits and integration by parts. In the case of practical problems too much stress cannot be laid upon the study of the determination of the constant of integration.

The suggestions that I have just made, while they are deliberate in spirit, are rather tentative as regards detail. The only way is for actual teachers of mathematics to make an inquiry into the practical uses of mathematics, and the easiest way for them to make this investigation in higher fields is for them to become students of engineering and mathematical physics. Something in the way of suggestion may be gotten

by a glance at Mellor's book on "Higher Mathematics for Students of Chemistry and Physics," published by Longmans & Co. The topics taken up are:

Differential Calculus,
Analytical Geometry,
Singularities of Functions,
Integral Calculus,
Infinite Series,
Solution of Numerical Equations,
Solution of Differential Equations,
Fourier's Theorem,
Theory of Errors and Probabilities,
Calculus of Variations,
Determinants and their Differentiation,
Formulæ,
Reference Tables.

Another book that will prove very suggestive is "Higher Mathematics," by Merriman & Woodward, published by Wiley & Sons. This is a book in which each article has been written by a pure mathematician to meet the needs of scientists. The topics taken up are:

Equations of Higher Degrees,
The Use of Determinants in Calculus,
Projective Geometry,
Hyperbolic Functions,
Harmonic Functions,
Functions of a Complex Variable,
Differential Equations,
Grassman's Space Analysis,
Vector Analysis and Quaternions,
Theory of Errors,
Mathematical History.

The topic of elliptic integrals appears to be omitted. The arrangement in the high school curriculum of the various topics I would recommend to be as follows:

First Year.....	First Term:	Elementary Algebra.
	Second Term:	Plane Geometry.
Second Year...	Third Term:	Advanced Algebra.
	Fourth Term:	Solid Geometry and Mensuration.

Third Year....	Fifth Term:	Trigonometry, Logarithms, Tables of all kinds, Methods of Approximation, the Slide Rule, and Surveying.
	Sixth Term:	Analytical Geometry.
Fourth Year...	Seventh Term:	Differential and Integral Calculus.
	Eighth Term:	Miscellaneous Problems, based as far as possible upon any every-day or scientific material with which the pupil has become familiar toward the end of his course, and presented so as to serve for a general review of his course in mathematics.

The first impression a mathematics teacher is likely to get from this suggestion to introduce analytical geometry and calculus, and to slash up the present traditional courses is that pupils will suffer indigestion from the former and obtain no culture from the latter. Personally, I do not doubt that a pupil in the fourth year who deliberately elects higher mathematics is just the kind of pupil ready to grasp the subject of calculus. As to culture, I feel confident that the useful topics retained may be arranged in a perfectly logical and progressive form. Adding to this the greater breadth involved, it is evident that an increase in culture may be obtained. While, then, mathematics may well for the present continue to be taught principally for culture, we shall not lose, but rather gain ground, by reorganizing our courses with a view to service. Let us strive to have culture and service go hand in hand.

STUYVESANT HIGH SCHOOL,
NEW YORK CITY.

"Give us, oh give us the man who sings at his work! Be his occupation what it may, he is the equal to any of those who follow the same pursuit in silent sullenness. He will do more in the same time—he will do it better—he will persevere longer. One is scarcely sensible of fatigue whilst he marches to music. The very stars are said to make harmony as they revolve in their spheres. Wondrous is the strength of cheerfulness, altogether past calculation its power of endurance. Efforts to be permanently useful must be uniformly joyous—a spirit all sunshine—graceful from very gladness—beautiful because bright."—THOMAS CARLYLE.

NEW BOOKS.

A Treatise on Differential Geometry. By LUTHER P. EISENHART. Boston: Ginn and Company. Pp. 474. $4.50.

It is the purpose of this book to introduce the student to the methods of differential geometry and to the theory of curves and surfaces developed thereby, to such an extent that he will be prepared to read the more extensive foreign treatises and journal articles. The reader is supposed to possess a knowledge of the calculus, elementary differential equations, and the elements of coördinate geometry of three dimensions. Hence the first half of the book may be used with seniors, and the remainder will constitute a full-year course for graduate students.

The method generally used is that of Gauss, common among German and Italian writers, but the kinematical method, frequently adopted in France, has been developed and applied where more feasible. This has been done not only because it furnishes the student with a powerful operator, but for the reason, also, that it develops geometrical thinking.

There are several hundred problems, some of which are direct applications of the accompanying sections, but many are theorems which might properly be established in a more extensive treatise. These have been inserted as an incentive to research and as preparation for larger problems.

The Calculus and its Applications. By ROBERT GORDON BLAINE. New York: D. Van Nostrand Company. Pp. 321. $1.50.

This book is compiled mainly from notes and examples prepared by the author for his classes. It is intended as an introduction to existing standard works on the subject. A large number of the problems are worked out for the sake of those students who desire to obtain a working acquaintance with the subject with little aid from a teacher. The examples given seem to be well selected and to cover the field of the various applications of the subject. It should prove a useful book for a first course.

Constructive Drawing. Revised edition. By HERMAN HANSTEIN. Chicago: Keuffel & Esser Company.

This book consists of about 140 problems and covers the work of the first year's course in the Chicago high schools and in the drawing department of the Chicago Mechanics Institute. The author's long experience has enabled him to select problems of practical importance to those who follow architecture, mechanical and engineering vocations, as well as problems which are indispensible to manufacturing and industrial pursuits.

Diary and Time Saver for 1910. Chicago: Laird and Lee.

This is one of the most useful diaries we have seen.

Exercises in Geometry. By GRACE LAWRENCE EDGETT. Boston: D. C. Heath & Co. Pp. 81.

The exercises in this book have been arranged in groups in such an order that it is in the nature of an appendix to the Harvard "Syllabus of Propositions in Geometry." They have been collected from various standard texts, from college entrance examinations and some have been evolved by the author.

Plane Trigonometry. By EDWARD R. ROBBINS. New York: American Book Company. Pp. 166. 60 cents.

This book is intended for high school and college preparatory courses. It is illustrated in the usual manner, but the diagrams are more than usually clear-cut and elucidating. No special tables are furnished, though the chapter on logarithms explains the use of tables in general.

The work is sound and teachable, and is written in clear and concise language, in a style that makes it easy for the beginner. Immediately after each principle has been proved, it is applied first in illustrative examples, and then further impressed by numerous exercises. All irrelevant and extraneous matter is excluded, thus giving greater emphasis to universal rules and formulas. Due emphasis is given to the theoretical as well as to the practical applications of the science. The number of examples, both concrete and abstract, is far in excess of those in other books on the market.

Plane Geometry Developed by the Syllabus Method. By EUGENE RANDOLPH SMITH. New York: American Book Company. Pp. 192. 75 cents.

This book has grown naturally from ten years' class work, and reflects the growing opinion among teachers that the proofs of geometry should be worked out by the pupils, rather than merely memorized. The list of theorems is sufficient for any college entrance examination. The laws of logic used in plane geometry are first stated in the most usable manner. The methods of discovering proofs are reduced to as few kinds as possible, and the definitions and axioms are given in quite complete form. The theorems are clearly stated, and the pupil is taught to discover the proofs by the application of his reasoning powers. The propositions are classified under heads suited to practical application to the work following. The exercises are numerous and helpful. This method stimulates the pupil to think to the limit of his ability, lays emphasis upon accuracy in the work, encourages originality, and develops individuality.

NOTES AND NEWS

THE next meeting of the association will be held at George Washington University, Washington, D. C., on Saturday, March 26. A good program is assured, as there will be some excellent papers, and the syllabus on advanced algebra will be presented. It is hoped that the members will attend in good numbers so as to make our first Washington meeting a success.

AT the meeting of The American Federation of Teachers of the Mathematical and the Natural Sciences at Boston during the holidays, some important business was transacted. The two things that most nearly concern teachers of mathematics were the report of the College Entrance Requirements Committee, which made special mention of the great variation in the entrance requirements of the different colleges in advanced algebra, and the appointment of a committee of one member from each of the allied associations composed in part or wholly of teachers of mathematics, to consider the advisability of having an official organ for teachers of mathematics. Dr. W. H. Metzler is the representative of this association on that committee.

The Committee on Syllabus in Geometry reported that work was well under way. The Committee has been divided into three sub-committees, one on logical considerations, one on lists of basal theorems and one on exercises and applications. The committee expects to have its work completed during the coming year.

The officers elected by the federation council for the ensuing year are:

President: C. R. Mann, University of Chicago.

Secretary-Treasurer: Eugene R. Smith, Polytechnic Preparatory School, Brooklyn.

Members of the Executive Committee: I. N. Mitchell, J. T. Rorer, Wilhelm Segerblom.

There are now fourteen associations allied in the federation.

WILL the members be kind enough to remember that communications from the officers asking for an answer should not be allowed to wait indefinitely. The work of carrying on an association of this size is not inconsiderable, and a little coöperation on the part of members would be greatly appreciated. The officers have to write to different members almost daily, and a little delay on the part of each correspondent becomes an almost unbearable burden to us who are already very busy.

THE secretary would appreciate suggestions from any members who have material on hand for papers to be read at the meetings. Also, any suggestions as to topics for discussion, work to be undertaken, or anything else that may be of value in the work of the association should be sent to him. Remember that it is your association, and that each of you is to some extent responsible for its continued success and for its value to you and to your fellow teachers.

EUGENE R. SMITH,
Secretary.

COLLEGE ALGEBRA

By WILLIAM H. METZLER, Ph.D., EDWARD D. ROE, Jr., Ph.D., and WARREN G. BULLARD, Ph.D., Professors of Mathematics in Syracuse University. 455 pages. Price, $1.50.

JUST PUBLISHED

A new edition containing in addition to the original text, a section of 94 pages on **INTRODUCTORY ALGEBRA**, covering the subject from the beginning through simultaneous quadratic equations.

MURRAY'S MATHEMATICS

Text-Books by D. A. MURRAY, Ph.D. (Johns Hopkins), Professor of Applied Mathematics in McGill University.

Differential Equations	**$1.90**
Differential and Integral Calculus	**2.00**
Infinitesimal Calculus	**2.00**
Logarithmic and Trigonometric Tables Four and Five Place Tables	**.50**
Plane Trigonometry	**.80**
Plane Trigonometry with Tables	**1.00**
Plane and Spherical Trigonometry	**1.00**
Plane and Spherical Trigonometry with Tables	**1.20**
Spherical Trigonometry	**.80**
Essentials of Trigonometry	**.80**

LONGMANS, GREEN, & CO., Publishers,
FOURTH AVENUE & 30TH STREET, NEW YORK

THE annual dinner of the Philadelphia Section will take place Saturday, April 9. In addition to several local speakers, representing the colleges and schools, Prof. H. B. Fine, of Princeton, has been invited to speak.

DR. E. S. CRAWLEY, of the University of Pennsylvania, has been appointed chairman of a committee to consider the character of logarithmic tables to be used in college entrance examinations. The committee was requested by the College Entrance Examination Board and appointed by the American Federation.

NEW MEMBERS.

S. BARNES, CAROLINE F., A.B.; High School, Watkins, N. Y.
S. BOND, MYRTLE GRACE, A.B.; 28 Pleasant St., Potsdam, N. Y.
CLARKE, MARTHA E.; 116 N. Caroline Ave., Washington, D. C.
S. COBANE, MARY ELIZABETH; High School, Skaneateles, N. Y.
S. COLLISTER, NORTON C., B.S.; High School, Johnstown, N. Y. 8 N. Melcher St.
S. CONIME, ELMER F., Ph.B.; Little York, N. Y.
N. Y. DANIELS, ELINOR LATHROP, A.B.; 72 E. 127th St., New York City.

YOUNG & JACKSON'S ALGEBRAS

JUST PUBLISHED

YOUNG & JACKSON'S SECOND COURSE IN ALGEBRA

A continuance of **YOUNG & JACKSON'S FIRST COURSE,** for the use of pupils who have done a year's work in Algebra.

This book is intended to cover a full second year's work.

CONVENIENT | **The Different Editions of these Books are:** | **UP-TO-DATE**

YOUNG & JACKSON'S ELEMENTARY ALGEBRA $1.12
YOUNG & JACKSON'S FIRST COURSE . $ .95
YOUNG & JACKSON'S SECOND COURSE .70

D. APPLETON & COMPANY

29-35 West 32d Street NEW YORK CITY

S. EVANS, W. E., B.Pd.; High School, Gardenville, N. Y.
S. FRASER, ETTA EVELYN, A.B.; High School, Brushton, N. Y.
S. HOWE, ANNA MAYME, A.B.; Rome Academy, Rome, N. Y. 215 N. Washington St.
S. KELLY, LYRA FOOTE, A.B.; Utica Free Academy, Utica, N. Y. 12 Scott St.
S. KENT, ANSON E., A.B.; Chautaqua, N. Y.
S. KNOX, ARTHUR E.; High School, Bingham, N. Y. Conklin, N. Y.
S. LAWRENCE, MORTON ROBERT; Skaneateles, N. Y.
S. METZLER, GEORGE FREDERICK, Ph.D.; Syracuse University, Syracuse, N. Y.
NEARING, ELENA P.; 8 Ash St., Flushing, N. J.
S. NORTHCOTT, JOHN ARTHUR, A.M.; Syracuse University, Syracuse, N. Y. 729 Ostrom Ave.
S. ROBSON, JENNIE, B.Pd.; 17 N. Perry St., Johnstown, N. Y.
S. STEWART, ALBERT BURNS, A.M.; Colgate University, Hamilton, N. Y.
WHITE, MARY ELIZABETH, A.B.; Miss Craven's School for Girls, Newark, N. J. 1008 Broad St.

WE RECOMMEND TEACHERS

Our system of getting information concerning teachers and positions is complete and accurate. We take special pleasure in sending a teacher to just the right kind of a position. All our business is done in this way—through careful investigation and discriminating recommendation. Usually one carefully selected candidate is notified of each vacancy, and that one fully recommended. This is the kind of service you want. We shall be glad to help you. Send for circulars.

THE REED TEACHERS' AGENCY

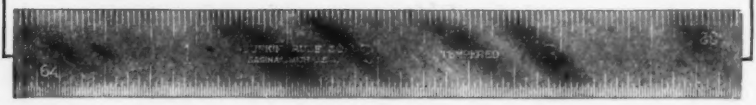
H. E. REED, Mgr.

639-641 University Block, SYRACUSE, N. Y.

LUFKIN STEEL RULES AND MEASURING TAPES

are especially useful and convenient for all classes of mathematical drawings and other scientific work. Complete catalog sent on request.

THE LUFKIN RULE CO. Saginaw, Mich., U. S. A.



NEWEST BOOKS

Wells's First Course in Algebra

A one-year course, with abundant well-graded problems, clear-cut proofs, and unique features in method and order. The work in graphs is illustrated in colored diagram. Half-leather; flexible. Price, $1.00.

Wells's Second Course in Algebra

For classes that have already had a one-year course. Reviews the elements and completes the preparation for entrance to college or technical school. Half-leather; flexible. Price, $1.00.

Wells's New Geometry

For several years educators have been attempting the solution of the geometry question. The old line books afford too little development to the pupil. The radical books are too radical. Professor Wells has been studying this subject for some time and now offers the ideal geometry. Plane and Solid, $1.25.

Wells's Algebra for Secondary Schools

This book provides a course for a year and a half. The full treatment of factoring, the illustrative use in the problems of scientific formulæ, the treatment of the graph, and the clear and rigorous demonstrations fit the book for successful use in the best schools. $1.20.

Wells's Text Book in Algebra

The TEXT-BOOK contains all of the chapters in Wells's Algebra for Secondary Schools, together with six chapters upon more advanced topics. It meets the maximum requirement in algebra for college entrance and is well suited to the needs of the first-year class in many colleges and technical schools. $1.40.

Nichols's Analytic Geometry

The revised edition that has just been published makes this book of exceptional value for introductory courses in analytics. $1.25.

Osborne's Differential and Integral Calculus

The revised and enlarged edition of Osborne's Calculus has met with exceptional favor. The work represents the happy combination of theory and application that is in demand both in general courses and in engineering courses. $2.00.

Our special catalogue of Mathematics for High Schools and Colleges will be mailed free to any address upon request.

D. C. HEATH & CO.

BOSTON NEW YORK CHICAGO

THE UNIVERSITY OF WISCONSIN

Courses in Arts, Sciences, Law, Engineering, Agriculture (eight new courses), Medicine, Education, Moral Education, Manual Training, Design, Physical Training, Public School Music. Regular graduate and undergraduate and special courses, for men and women. Full credit toward degrees. **MASTER'S DEGREE** in three summer sessions with home study.

SUMMER SESSION, 1910

Six Weeks---June 27 to August 5 **(Law, 10 Weeks---June 27 to September 2)**

Located in Madison, the beautiful, the city of the four lakes. 130 miles northwest of Chicago. Except in Law (fee $25) the fee is $15.

One Fee for One or All Courses. Write for illustrated bulletin

University of Wisconsin **Madison, Wis.**

How many names of new members will you send to the Secretary before March 26?

HAS the library of your school sent their subscription to *The Mathematics Teacher* yet?

IF GOING TO

WASHINGTON, D. C.

Write for Handsome Descriptive Booklet and Map

HOTEL RICHMOND

17th and H Streets N. W.

A Model Hotel Conducted for Your Comfort

Location and size: Around the corner from the White House. Direct street car route to palatial Union Station. 100 Rooms, 50 Baths.

Plans, rates and features; European, $1.50 per day upward; with Bath $2.50 upward. American, $3.00 per day upward; with Bath, $4.00 upward. Club breakfast 20 to 75c. Table d'Hote, breakfast $1.00; Luncheon 50c; Dinner $1.00; Music.

CLIFFORD M. LEWIS, Prop.

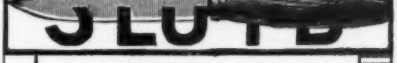
SUMMER SEASON; The American Luzerne in the Adirondack foot hills. Wayside Inn and Cottages on beautiful Lake Luzerne, Warren Co., N. Y. Open June 26 to Oct. 1.

BOOKLET

Practical Instruction

Cannot be attained with poor tools. As pioneers in School Supplies we are prepared to furnish complete outfits and quote unusual discounts.

SLOYD



BENCHES, TOOLS AND SUPPLIES } ARE THE Best

Carving Tools

Send for Catalog. SINGLY AND IN SETS

CHANDLER & BARBER, BOSTON

60 YEARS' EXPERIENCE

PATENTS

TRADE MARKS
DESIGNS
COPYRIGHTS &C.

Anyone sending a sketch and description may quickly ascertain our opinion free whether an invention is probably patentable. Communications strictly confidential. HANDBOOK on Patents sent free. Oldest agency for securing patents.

Patents taken through Munn & Co. receive *special notice*, without charge, in the

Scientific American.

A handsomely illustrated weekly. Largest circulation of any scientific journal. Terms, $3 a year; four months, $1. Sold by all newsdealers.

MUNN & Co. 361 Broadway, **New York**

Branch Office, 625 F St., Washington, D. C.

New from Cover to Cover

WEBSTER'S NEW INTERNATIONAL DICTIONARY

JUST ISSUED. Ed. in Chief, Dr. W. T. Harris, former U. S. Com. of Education. ▱ General Information Practically Doubled. ▱ Divided Page: Important Words Above, Less Important Below. ▱ Contains More Information of Interest to More People Than Any Other Dictionary.

2700 PAGES. 6000 ILLUSTRATIONS. 400,000 WORDS AND PHRASES.

GET THE BEST in Scholarship, Convenience, Authority, Utility.

Write for Specimen Pages to

G. & C. MERRIAM CO., Publishers, Springfield, Mass.

You will do us a favor to mention this publication.

Little Differences

indicate the boundary between the ordinary and the superior. The designs of our jewelry are a little more exclusive than any others, the workmanship a little finer, the quality of the gold and the gems a little better—little differences you will certainly appreciate.

ROLLED GOLD PINS, LOCKETS and SLEEVE LINKS
in quaint new styles for the economical purse.

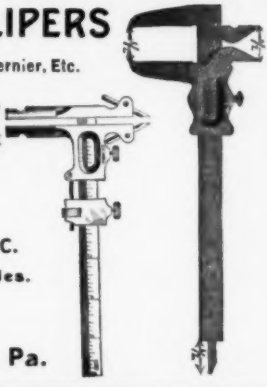
STETSON & CROUSE

127 South Salina Street, SYRACUSE, N. Y.

COLUMBIA CALIPERS

Graduated in Inches, Metric, Vernier, Etc.

Thousands in use. Many for twenty years, giving excellent service.



MICROMETERS,
SPHEROMETERS, ETC.
for Physical Laboratories.

Send for catalog and special prices

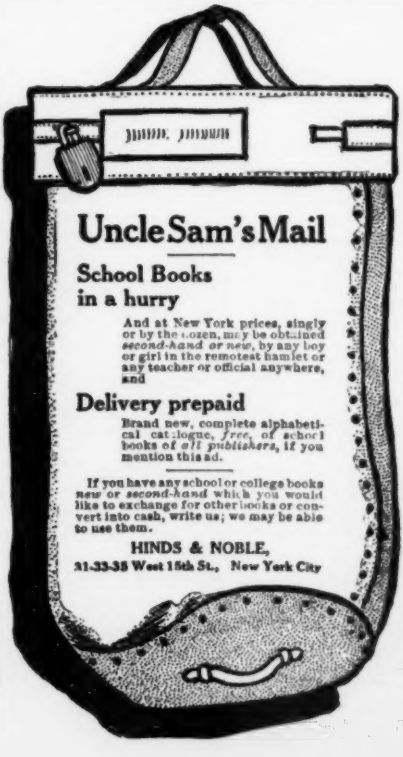
E. G. SMITH, Columbia, Pa.

Stanley Tools

STANDARD THE WORLD OVER

EVERY TOOL GUARANTEED

Stanley Rule & Level Co.
NEW BRITAIN, CONN. U.S.A.



Uncle Sam's Mail

School Books in a hurry

And at New York prices, singly or by the dozen, may be obtained *second-hand or new*, by any boy or girl in the remotest hamlet or any teacher or official anywhere, and

Delivery prepaid

Brand new, complete alphabetical catalogue, *free*, of school books of *all publishers*, if you mention this ad.

If you have any school or college books *new* or *second-hand* which you would like to exchange for other books or convert into cash, write us; we may be able to use them.

HINDS & NOBLE,
31-33-35 West 15th St., New York City

THE NEW ERA PRINTING COMPANY

LANCASTER, PA.

Is prepared to execute in first-class and satisfactory manner all kinds of printing and electrotyping. Particular attention given to the work of Schools, Colleges, Universities, and Public Institutions.

Books, Periodicals

Technical and Scientific Publications

Monographs, Theses, Catalogues

Announcements, Reports, etc.

All Kinds of Commercial Work

(*Printers of the Bulletin and Transactions of the American Mathematical Society, etc., etc.*)

Publishers will find our product ranking with the best in workmanship and material, at satisfactory prices. Our imprint may be found on a number of high-class Technical and Scientific Books and Periodicals. Correspondence solicited. Estimates furnished.

THE NEW ERA PRINTING COMPANY

RARA ARITHMETICA

A catalogue of arithmetics printed before 1601 with a description of those in the library of George A. Plimpton, Esq.

By David Eugene Smith

Professor of Mathematics in Teachers College, Columbia University

A necessary book of reference for all bibliophiles, librarians, students of education and teachers of elementary and secondary mathematics.

Student's Edition $4.50

The most elaborate bibliography of arithmetic yet attempted, Rara Arithmetica contains a description of most of the important arithmetics of the formative period in the modern history of the subject.

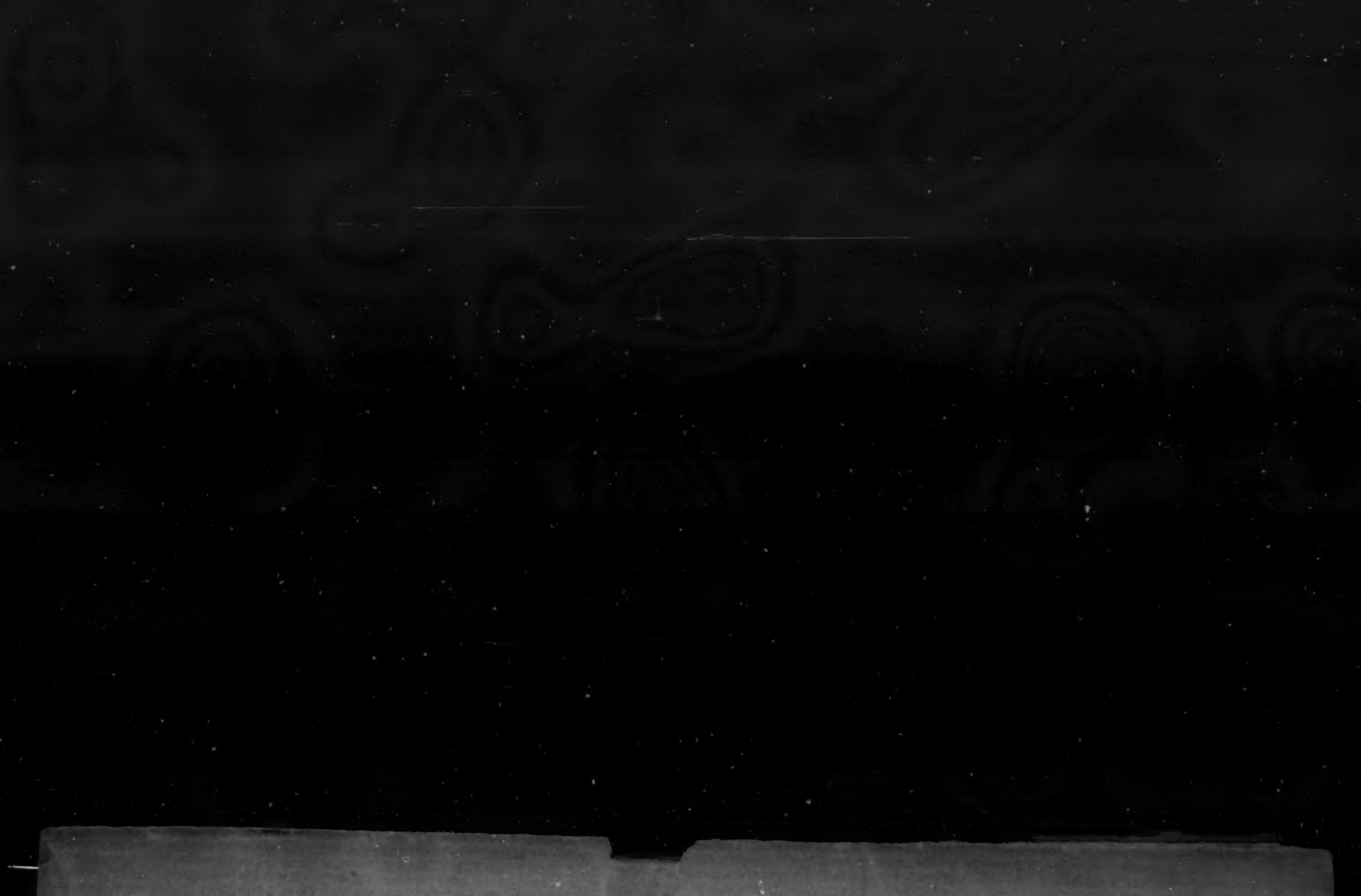
The illustrations are particularly valuable. There are more than two hundred and fifty interesting facsimiles, including title pages and illustrations from rare first editions.

This is a contribution to mathematical literature which will be welcomed by a large body of mathematicians.—W. F. Osgood, LL.D., Professor of Mathematics in Harvard University.

It is of very high value for scholars and for students in history and mathematics.
—K. Goldziher, Professor in the National Paedagogium, Budapest.

The Rara Arithmetica is a remarkable work of great importance, and a beautiful piece of bookmaking.—Marquis Guccia, Professor of Mathematics in the University of Palermo.

A masterpiece of American scholarship.
—F. N. Cole, Professor of Mathematics in Columbia University.

GINN AND COMPANY, Publishers

Boston	New York	Chicago	London
Atlanta	Dallas	Columbus	San Francisco